Chapter 1

LORENTZ/POINCARE INVARIANCE

1.1 The Lorentz Algebra

The requirement of relativistic invariance on any fundamental physical system amounts to invariance under *Lorentz Transformations*. These transformations include *boosts* and spatial rotations and make up the six-parameter *Lorentz Group*. Lorentz invariance corresponds to the *isotropy of spacetime*. On the other hand the *homogeneity of spacetime* corresponds to invariance under *spacetime translations* or *Poincaré Transformations*. The combined set of these transformations makes up a ten-parameter group the *Lorentz-Poincaré Group or Inhomogeneous Lorentz Group*.

A *Lorentz transformation* of spacetime coordinates[1]

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu}\, x^{\nu}, \tag{1.1}$$

defined by the requirement that it leaves[2] $s^2 = x^{\mu}\, x^{\nu}\, g_{\mu\nu}$ invariant, has to obey

$$\Lambda^{\mu}_{\ \sigma}\, g^{\sigma\rho}\, \Lambda^{\nu}_{\ \rho} = g^{\mu\nu}\,. \tag{1.2}$$

For an *infinitesimal Lorentz transformation*

$$\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\ \nu}\,, \tag{1.3}$$

with $\omega << 1$, the relations (1.1) and (1.2) lead to

$$\delta x^{\mu} \equiv x'^{\mu} - x^{\mu} = \omega^{\mu}_{\ \nu}\, x^{\nu}, \quad \omega_{\mu\nu} = -\omega_{\nu\mu}\,. \tag{1.4}$$

Note that only the so called *Proper Lorentz Transformations*, i.e. those that have $det(\Lambda) = +1$, can be decomposed continuously into infinitesimal transformations as in (1.2). In addition to these *continuous* transformations, we also have two *Discrete Lorentz Transformations*, namely the *Spatial Reflection (Parity)*

$$\mathcal{P} : (x^0,\, x^i) \to (x^0,\, -x^i) \Longrightarrow \Lambda_P = Diag(1,\, -1,\, -1,\, -1) \tag{1.5}$$

and the *Time Reversal*

$$\mathcal{T} : (x^0,\, x^i) \to (-x^0,\, x^i) \Longrightarrow \Lambda_T = Diag(-1,\, 1,\, 1,\, 1)\,. \tag{1.6}$$

Although invariance under proper Lorentz transformations is an exact symmetry of the world, this cannot be said for parity and

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[1] All physical quantities are classified according to their transformation properties under Lorentz Transformations. A (contravariant) four-vector, by definition, transforms exactly as the spacetime position four-vector x^{μ}

$$\mathcal{A}^{\mu} \to \mathcal{A}'^{\mu} = \Lambda^{\mu}_{\ \nu}\, \mathcal{A}^{\nu}\,.$$

Tensor quantities transform in terms of more than one Lorentz matrices as

$$\mathcal{F}'^{\mu\nu} = \Lambda^{\mu}_{\ \alpha}\, \Lambda^{\nu}_{\ \beta}\, \mathcal{F}^{\alpha\beta}\,.$$

[2] Throughout these notes we have chosen the metric to be $g_{\mu\nu} = Diag(1,\, -1,\, -1,\, -1)$.

time reversal, or combinations of them with a proper transformation, which are only approximate symmetries. Thus, the set of all Lorentz transformations is divided into four subsets or, equivalently, the *Lorentz Group* consists of four disconnected parts. These are[3] the *Proper-Orthocronous*, or simply, proper Lorentz transformations $\mathbf{L}_{\uparrow}^{(+)}$ with $det(\Lambda) = +1$, $\Lambda^0_{\ 0} \geq 1$, the *Proper, Non-Orthochronous* ones $\mathbf{L}_{\downarrow}^{(+)}$ with $det(\Lambda) = +1$, $\Lambda^0_{\ 0} \leq -1$, the *Improper-Orthocronous* ones $\mathbf{L}_{\uparrow}^{(-)}$ with $det(\Lambda) = -1$, $\Lambda^0_{\ 0} \geq 1$ and the *Improper, Non-Orthocronous* ones $\mathbf{L}_{\downarrow}^{(-)}$ with $det(\Lambda) = -1$, $\Lambda^0_{\ 0} \leq -1$.

Note that a generic Lorentz transformation Λ can be written as

$$\Lambda = e^{\omega} \tag{1.7}$$

in terms of a matrix $\omega^{\mu}_{\ \nu}$ which is not necessarily infinitesimal. Then, from (1.2) we obtain

$$\Lambda^{\perp} g \Lambda = g \Longrightarrow e^{\omega^{\perp}} g\, e^{\omega} = g \Longrightarrow g\, e^{\omega^{\perp}} g\, e^{\omega} = 1\,.$$

The last equation means that

$$g \Lambda^{\perp} g = \Lambda^{-1}$$

or

$$g\, e^{\omega^{\perp}} g = e^{-\omega} \Longrightarrow e^{g\omega^{\perp}g} = e^{-\omega} \Longrightarrow g\omega^{\perp} g = -\omega\,.$$

Since, $(\omega^{\perp})_{\nu}^{\ \mu} = \omega^{\mu}_{\ \nu}$, this equation implies

$$\omega_{\nu}^{\ \mu} = -\omega^{\mu}_{\ \nu}\,. \tag{1.8}$$

Thus, the matrix ω is antisymmetric.

Let us next introduce the six matrices $J^{\mu\nu}$ defined by their elements

$$(J^{\mu\nu})^{\alpha}_{\ \beta} = i\left(g^{\mu\alpha} g^{\nu}_{\beta} - g^{\mu}_{\beta} g^{\nu\alpha} \right)\,. \tag{1.9}$$

[3] Note that

$$\Lambda^{\mu}_{\ 0} g_{\mu\nu} \Lambda^{\nu}_{\ 0} = g_{00} = 1 \Longrightarrow \left(\Lambda^0_{\ 0}\right)^2 = 1 + \sum_{i=1}^{3} \left(\Lambda^i_{\ 0}\right)^2 \geq 1\,.$$

In terms of the J's we have the identity

$$\omega^{\mu}_{\ \nu} \, = \, -\frac{i}{2}\omega_{\alpha\beta}\,\left(J^{\alpha\beta}\right)^{\mu}_{\ \nu}\,. \tag{1.10}$$

Thus, the generic Lorentz transformation can also be written as

$$\Lambda \, = \, e^{-\frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}}\,. \tag{1.11}$$

The corresponding infinitesimal transformation reads

$$\delta x^{\mu} \, = \, \omega^{\mu}_{\ \nu}\, x^{\nu} \, = \, -\frac{i}{2}\omega_{\alpha\beta}\,\left(J^{\alpha\beta}\right)^{\mu}_{\ \nu}\, x^{\nu}\,. \tag{1.12}$$

The matrices $J^{\mu\nu}$ are the six *Generators* of the *Lorentz Group* and they satisfy the *Algebra*[4]

$$[J_{\mu\nu},\, J_{\rho\sigma}] = ig_{\nu\rho}J_{\mu\sigma} - ig_{\mu\rho}J_{\nu\sigma} - ig_{\nu\sigma}J_{\mu\rho} + ig_{\mu\sigma}J_{\nu\rho}\,. \tag{1.13}$$

Note that the group elements (1.11) are not unitary since, although J^{ij} are Hermitean, J^{0i} are anti-Hermitean.

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[4]**Representations of the Lorentz group.**
Let us introduce the following combinations of the generators

$$J^{(1)}_i \,\equiv\, \frac{1}{4}\epsilon_{ijk}J_{jk} \,-\, \frac{i}{2}J_{0i}, \quad J^{(2)}_i \,\equiv\, \frac{1}{4}\epsilon_{ijk}J_{jk} \,+\, \frac{i}{2}J_{0i}\,.$$

With some tedious but straightforward computation it can be shown that these six operators satisfy the following commutation relations

$$\left[\,J^{(1)}_i,\, J^{(1)}_j\,\right] \,=\, i\epsilon_{ijk}\, J^{(1)}_k, \quad \left[\,J^{(2)}_i,\, J^{(2)}_j\,\right] \,=\, i\epsilon_{ijk}\, J^{(2)}_k, \quad \left[\,J^{(1)}_i,\, J^{(2)}_j\,\right] \,=\, 0$$

The fact that the Lorentz Algebra $so(3,1)$ (Lie groups are usually denoted by capitals ($SO(3,\,1)$, $.SO(3)$, $SU(2)$, ...) while the corresponding Algebras with lower case letters ($so(3,1)$, $so(3)$, $su(2)$, ...) corresponds to the Algebra of two independent angular momenta ($so(3) \oplus so(3)$)) is expressed mathematically as an *isomorphism*

$$so(3,\,1) \,\simeq\, so(3) \oplus so(3) \,\simeq\, su(2) \oplus su(2)\,.$$

This correspondence does not mean that the groups are the same. The Algebra determines the group properties only for infinitesimal transformations or, equivalently, in the vicinity of the identity. This is clear if one considers the matrix representation of the generators. Then, a $SO(3,1)$ transformation is non-unitary, in contrast to the $SU(2)$ transformations which manifestly unitary.

It is straightforward now to use the above correspondence and the elementary theory of angular momentum in order to classify each representation with the pair of angular momentum quantum numbers $(j_1,\, j_2)$, each of which can have values

$$j_1,\, j_2 \,=\, 0,\, 1/2,\, 1,\, 3/2,\, 2,\, \ldots\,.$$

The multiplicity (*dimensionality*) of each representation is $(2j_1+1)(2j_2+1)$.

1.1.1 Representation of the Lorentz Algebra in terms of differential operators

In addition to the above expressions of finite or infinitesimal Lorentz transformations of the spacetime coordinates as 4×4 matrices, it is possible to represent them as differential operators. Considering an infinitesimal transformation $\delta x^{\mu} = \omega^{\mu}_{\ \alpha}\, x^{\alpha}$, this goes as follows

$$\delta x^{\mu} = \omega^{\beta}_{\ \alpha}\, \delta^{\mu}_{\beta}\, x^{\alpha} = \omega^{\beta}_{\ \alpha}\, \partial_{\beta} x^{\mu}\, x^{\alpha} = \omega^{\beta}_{\ \alpha}\, x^{\alpha}\, \partial_{\beta} x^{\mu} = \omega_{\beta\alpha}\, x^{\alpha}\, \partial^{\beta} x^{\mu}$$

$$= \frac{1}{2}\left(\omega_{\beta\alpha}\, x^{\alpha}\, \partial^{\beta} + \omega_{\alpha\beta}\, x^{\beta}\, \partial^{\alpha}\right) x^{\mu} = \frac{1}{2}\omega_{\alpha\beta}\left(x^{\beta}\, \partial^{\alpha} - x^{\alpha}\partial^{\beta}\right) x^{\mu}$$

or

$$\delta x^{\mu} = \frac{i}{2}\omega_{\alpha\beta}\, \mathcal{L}^{\alpha\beta}\, x^{\mu}\,, \tag{1.14}$$

where the quantities $\mathcal{L}^{\alpha\beta}$, defined by

$$\mathcal{L}^{\mu\nu} \equiv i\,(\, x^{\mu}\, \partial^{\nu} - x^{\nu}\, \partial^{\mu}\,)\,, \tag{1.15}$$

represent the six *Generators* of the *Lorentz Group* as differential operators in the case of the spacetime coordinate transformations. In the form (1.14) the infinitesimal transformation is expressed as resulting from the action of an operator

$$\hat{L}(\omega) \equiv \frac{i}{2}\omega_{\alpha\beta}\, \mathcal{L}^{\alpha\beta} \implies \delta x^{\mu} = \hat{L}(\omega)\, x^{\mu}\,. \tag{1.16}$$

The generators (1.15) are in one to one correspondence with the matrices $J^{\mu\nu}$ introduced previously. The generators satisfy the $SO(3,1)$ *Lie Algebra*

$$[\mathcal{L}_{\mu\nu}, \mathcal{L}_{\rho\sigma}] = ig_{\nu\rho}\mathcal{L}_{\mu\sigma} - ig_{\mu\rho}\mathcal{L}_{\nu\sigma} - ig_{\nu\sigma}\mathcal{L}_{\mu\rho} + ig_{\mu\sigma}\mathcal{L}_{\nu\rho}\,. \tag{1.17}$$

This is the same Algebra as (1.13).

This Algebra can also be expressed in terms of the operators (1.16). It amounts to the simple commutation relation

$$\left[\,\hat{L}(\omega_1),\, \hat{L}(\omega_2)\,\right] = -\hat{L}(\omega_3)\,, \tag{1.18}$$

where

$$\omega_3^{\mu\nu} \equiv [\,\omega_1,\, \omega_2\,]^{\mu\nu}\,. \tag{1.19}$$

1.2 Poincaré Transformations

Next we may consider spacetime translations

$$x^\mu \to x'^\mu = x^\mu + \alpha^\mu . \tag{1.20}$$

An infinitesimal Poincaré transformation is

$$\delta x^\mu = x'^\mu - x^\mu = \epsilon^\mu . \tag{1.21}$$

A differential operator representation of a Poincaré transformation can be readily derived writing (1.21) as

$$\delta x^\mu = \epsilon^\alpha \partial_\alpha x^\mu ,$$

which can be further written as

$$\delta x^\mu = i\epsilon^\alpha \mathcal{P}_\alpha x^\mu . \tag{1.22}$$

The quantities

$$\mathcal{P}_\mu \equiv -i\partial_\mu , \tag{1.23}$$

are the four *generators* of the *Poincaré Group* in the case of spacetime coordinate transformations. Again, an infinitesimal Poincaré transformation can be expressed in terms of an operator

$$\hat{T}(\epsilon) \equiv i\epsilon^\alpha \mathcal{P}_\alpha \Longrightarrow \delta x^\mu = \hat{T}(\epsilon)\, x^\mu . \tag{1.24}$$

The complete *Poincaré-Lorentz* Lie Algebra can be written down in terms of the ten generators $\mathcal{L}_{\mu\nu}$ and $\mathcal{P}_\mu = -i\partial_\mu$ as

$$\begin{aligned} &[\mathcal{L}_{\mu\nu}, \mathcal{L}_{\rho\sigma}] = ig_{\nu\rho}\mathcal{L}_{\mu\sigma} - ig_{\mu\rho}\mathcal{L}_{\nu\sigma} - ig_{\nu\sigma}\mathcal{L}_{\mu\rho} + ig_{\mu\sigma}\mathcal{L}_{\nu\rho} \\ &[\mathcal{L}_{\mu\nu}, \mathcal{P}_\rho] = -ig_{\mu\rho}\mathcal{P}_\nu + ig_{\nu\rho}\mathcal{P}_\mu \\ &[\mathcal{P}_\mu, \mathcal{P}_\nu] = 0 \end{aligned} \tag{1.25}$$

In analogy to (1.16) and (1.24) we can introduce operators expressing an infinitesimal combined Poincaré-Lorentz transformation

$$\hat{L}(\omega, \epsilon) \equiv \frac{i}{2}\omega_{\alpha\beta}\, \mathcal{L}^{\alpha\beta} + i\epsilon^\alpha \mathcal{P}_\alpha \Longrightarrow \delta x^\mu = \hat{L}(\omega, \epsilon)\, x^\mu . \tag{1.26}$$

It is not difficult to show that

$$\left[\,\hat{L}(\omega_1,\,\epsilon_1),\,\hat{L}(\omega_2,\,\epsilon_2)\,\right]\;=\;-\hat{L}(\omega_3,\,\epsilon_3)\,, \tag{1.27}$$

where

$$\omega_3^{\mu\nu}\;=\;[\,\omega_1,\,\omega_2]^{\mu\nu}\,,\qquad \epsilon_3^{\mu}\;=\;\epsilon_{1\nu}\,\omega_2^{\nu\mu}\,-\,\epsilon_{2\nu}\,\omega_1^{\nu\mu}\,. \tag{1.28}$$