Chapter 2

CLASSICAL FIELDS

A *field* is a physical object that is a spacetime function with well defined transformation properties under the Lorentz group. At the classical level, fields satisfy wave-like equations just like the electromagnetic (four-) vector potential in the absence of sources ($\partial^2 \mathcal{A}^\mu = 0$). Upon quantization, particles will come out as states, generated by the action of these fields on the *vacuum* state, transforming according to the representations of the Lorentz group.

2.1 Poincaré Transformations

The homogeneity of spacetime requires that any fundamental field should be translationally invariant. Thus, all fields $\phi(x)$ are assumed to be invariant under Poincaré translations. For any spacetime translation, we have

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu \Longrightarrow \phi'(x') = \phi(x)\,. \tag{2.1}$$

For an infinitesimal spacetime translation, we may introduce the purely functional change of the field at a given point as

$$\delta_0\phi(x) \equiv \phi'(x) - \phi(x) = \phi'(x' - \delta x) - \phi(x)$$

$$\approx \phi'(x') - \delta x^\mu \partial_\mu \phi'(x') - \phi(x) = -\delta x^\mu \partial_\mu \phi'(x') \approx -\delta x^\mu\, \partial_\mu \phi(x)$$

or

$$\delta_0\phi(x) = -i\epsilon^\mu \mathcal{P}_\mu \phi(x)\,. \tag{2.2}$$

2.2 Lorentz Transformations

Consider now the Lorentz transformation of a field $\phi(x)$

$$x^\mu \to x'^\mu \;=\; \Lambda^\mu_{\ \nu} x^\nu \quad \Rightarrow \quad \phi(x) \to \phi'(x')\,. \tag{2.3}$$

The infinitesimal variation[1]

$$\delta\phi(x) \equiv \phi'(x') - \phi(x) \tag{2.4}$$

is, in general, non-zero. For an infinitesimal Lorentz transformation

$$\delta x^\mu = \omega^\mu_\nu x^\nu = \frac{i}{2}\omega^{\rho\sigma}\mathcal{L}_{\rho\sigma}x^\mu \tag{2.5}$$

the field variation is

$$\delta\phi(x) = \phi'(x) - \phi(x) + \frac{i}{2}\omega^{\rho\sigma}\mathcal{L}_{\rho\sigma}\phi(x) \tag{2.6}$$

Introducing the variation

$$\delta_0\phi(x) = \phi'(x) - \phi(x) \tag{2.7}$$

we have

$$\delta\phi(x) = \delta_0\phi(x) + \frac{i}{2}\omega^{\rho\sigma}\mathcal{L}_{\rho\sigma}\phi(x) \tag{2.8}$$

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[1]Note that there are two kinds of variations. The *complete variation*

$$\delta\phi(x) \;\equiv\; \phi'(x') \;-\phi(x)$$

and the *purely functional variation*

$$\delta_0\phi(x) \;\equiv\; \phi'(x) \;-\phi(x)\,.$$

The two kinds of variations are related as

$$\delta\phi(x) \;=\; \delta_0\phi(x) \;+\; \delta x^\mu\,\partial_\mu\phi(x)\,.$$

For a Poincaré transformation $\delta\phi(x) = 0$ for any kind of field. Similarly, for a Lorentz transformation of a *scalar field*, $\delta\phi = 0$ as well.

2.3 Lorentz Transformation of a Scalar Field

A *Scalar Field* is, by definition, Lorentz-invariant

$$x'^{\mu} \,=\, \Lambda^{\mu}_{\ \nu}\, x^{\nu} \implies \phi'(x') \,=\, \phi(x)\,. \tag{2.9}$$

Thus, we also have

$$\phi'(x) \,=\, \phi(\Lambda^{-1}x)\,, \tag{2.10}$$

where $(\Lambda^{-1})_{\mu}^{\ \nu}$ is the inverse Lorentz transformation defined by $\Lambda^{\mu}_{\ \nu}(\Lambda^{-1})_{\rho}^{\ \nu} = \delta^{\mu}_{\rho}$. Thus, for an infinitesimal Lorentz transformation ($\Lambda^{\mu}_{\ \nu} \approx \delta^{\mu}_{\nu} + \omega^{\mu}_{\ \nu}$ and $(\Lambda^{-1})_{\mu}^{\ \nu} \approx \delta_{\mu}^{\nu} + \omega_{\mu}^{\ \nu}$), we shall have

$$\delta\phi(x) = \phi'(x') - \phi(x) = 0 \ \ \Rightarrow \ \delta_0\phi(x) = -\frac{i}{2}\omega^{\rho\sigma}\mathcal{L}_{\rho\sigma}\phi(x)\,. \tag{2.11}$$

Summarizing, for a scalar field, the corresponding variations under Lorentz transformations, are

$$\begin{aligned} \delta\phi(x) \,&=\, 0 \\ \delta_0\phi(x) \,&=\, -\tfrac{i}{2}\omega^{\rho\sigma}\mathcal{L}_{\rho\sigma}\phi(x) \end{aligned} \tag{2.12}$$

2.4 Lorentz Transformation of a Vector Field

The gradient of a scalar field is expected to transform as a vector field. Under an infinitesimal Lorentz transformation, we obtain the variation

$$\delta(\partial_\mu\phi(x)) = \partial'_\mu\phi'(x') - \partial_\mu\phi(x) = \partial'_\mu\phi(x) - \partial_\mu\phi(x) =$$

$$(\partial'_\mu x^\nu)\partial_\nu\phi(x) - \partial_\mu\phi(x) = \partial'_\mu(x'^\nu - \omega^{\nu}_{\ \rho}x^\rho)\partial_\nu\phi(x) - \partial_\mu\phi(x) \,=\, -\omega^{\nu}_{\ \mu}\partial_\nu\phi(x)$$

The last expression can be written in terms of the matrices

$$\left(J_{\rho\sigma}\right)^{\nu}_{\ \mu} \equiv i\left(g^{\nu}_{\rho}g_{\sigma\mu} - g^{\nu}_{\sigma}g_{\rho\mu}\right) \tag{2.13}$$

which satisfy the Lorentz Lie Algebra commutation relations. In terms of them, we have

$$\delta(\partial_\mu \phi(x)) = -\frac{i}{2}\omega^{\rho\sigma}\left(J_{\rho\sigma}\right)^{\nu}_{\ \mu}\partial_\nu\phi(x) \tag{2.14}$$

On the other hand, we can easily compute the restricted variation

$$\delta_0(\partial_\mu\phi(x)) = \delta(\partial_\mu\phi(x)) - \delta x^\nu\,\partial_\nu\partial_\mu\phi(x) =$$

$$-\frac{i}{2}\omega^{\rho\sigma}\left(J_{\rho\sigma}\right)^{\nu}_{\ \mu}\partial_\nu\phi(x) - \frac{i}{2}\omega^{\rho\sigma}\mathcal{L}_{\rho\sigma}\partial_\mu\phi(x)$$

which can be written as

$$\delta_0(\partial_\mu\phi(x)) = -\frac{i}{2}\omega^{\rho\sigma}\left(\mathcal{J}_{\rho\sigma}\right)^{\nu}_{\mu}\partial_\nu\phi(x) \tag{2.15}$$

where

$$\left(\mathcal{J}_{\rho\sigma}\right)^{\nu}_{\mu} = \mathcal{L}_{\rho\sigma}g^{\nu}_{\mu} + \left(J_{\rho\sigma}\right)^{\nu}_{\mu} \tag{2.16}$$

Equations (2.14) and (2.15) define the transformation properties of a general *Vector Field* $A_\mu(x)$

$$\begin{aligned} \delta A_\mu(x) &= -\tfrac{i}{2}\omega^{\rho\sigma}\left(\mathcal{S}_{\rho\sigma}\right)^{\nu}_{\mu}A_\nu(x) \\ \delta_0 A_\mu(x) &= -\tfrac{i}{2}\omega^{\rho\sigma}\left(\mathcal{J}_{\rho\sigma}\right)^{\nu}_{\mu}A_\nu(x) \end{aligned} \tag{2.17}$$

The objects (2.16) are the Lorentz generators for a vector, the second piece (J) corresponding to the *spin* of the vector field.

The corresponding finite Lorentz Transformations read

$$\begin{aligned} A'_\mu(x') &= \left(\Lambda^{-1}\right)_{\mu}^{\ \nu}A_\nu(x), \quad & A'_\mu(x) &= \left(\Lambda^{-1}\right)_{\mu}^{\ \nu}A_\nu(\Lambda^{-1}x) \\ A'^{\mu}(x') &= \Lambda^{\mu}_{\ \nu}A^\nu(x), & A'^{\mu}(x) &= \Lambda^{\mu}_{\ \nu}A^\nu(\Lambda^{-1}x) \end{aligned} \tag{2.18}$$

2.5 Spinors and Spinor Fields

It is possible to have fields with values that are not complex or real numbers but *Grassmann numbers*, i.e. quantities that anticommute

$$\theta_1\,\theta_2 = -\theta_2\,\theta_1\,. \tag{2.19}$$

It is obvious that the square of a Grassmann number always vanishes, i.e.

$$\theta^2 \,=\, 0\,. \tag{2.20}$$

A physical object that is Grassman-valued should have at least two components, so that quadratic terms are possible. Let us introduce such an object ψ_a. It will be called a two-component *left-handed Weyl spinor* and it will transform under Lorentz transformations as

$$\psi'_a \,=\, (\Lambda_L)_{ab}\,\psi_b\,. \tag{2.21}$$

Its square can be defined as

$$\psi^2 \,\equiv\, \psi_a\,\epsilon_{ab}\,\psi_b \,=\, -\psi_b\,\epsilon_{ab}\,\psi_a\,, \tag{2.22}$$

where $\epsilon_{ab} = -\epsilon_{ba}$ and $\epsilon_{12} = +1$. The Lorentz transformation matrix Λ_L of a left-handed Weyl spinor can now be determined from the requirement of Lorentz invariance of ψ^2. Since any 2×2 complex matrix can be written in terms of the Pauli matrices and the unit matrix, the matrix Λ_L can be put in the form

$$\Lambda_L \,=\, c_0\mathbf{1} \,+\, c_j\sigma_j\,. \tag{2.23}$$

Demanding that a Lorentz transformation should leave ψ^2 invariant, we obtain

$$\Lambda_L^{\perp}\,\epsilon\,\Lambda_L \,=\, \epsilon \implies \Lambda_L^{\perp}\,\sigma_2\,\Lambda_L \,=\, \sigma_2$$

or

$$\left(c_0\mathbf{1} \,+\, c_j\sigma_j^{\perp}\right)\sigma_2\left(c_0\mathbf{1} \,+\, c_i\sigma_i\right) \,=\, \sigma_2$$

or

$$\left(c_0\mathbf{1} \,-\, c_j\sigma_2\sigma_j\sigma_2\right)\sigma_2\left(c_0\mathbf{1} \,+\, c_i\sigma_i\right) \,=\, \sigma_2$$

or

$$c_0^2\mathbf{1} \,-\, c_ic_j\sigma_2\sigma_i\sigma_j \,=\, \sigma_2 \implies c_0^2 - c_j^2 \,=\, 1\,.$$

The last constraint allows Λ to be written as[2]

$$\Lambda_L \,=\, e^{\frac{i}{2}(\vec{\omega}\,+\,i\vec{\nu})\cdot\vec{\sigma}}\,. \tag{2.24}$$

For two different left-handed Weyl spinors ψ_L and χ_L, it is clear that the bilinear

$$\psi_L\,\chi_L \,+\, h.c. \,\equiv\, \psi_L\,\epsilon\,\chi_L \,+\, h.c.$$

is invariant under Lorentz transformations.

In a similar way we may introduce a *right-handed Weyl spinor*

$$\chi_R \,=\, i\sigma_2\,\psi_L^* \tag{2.25}$$

transforming in terms of[3]

$$\Lambda_R \,=\, e^{\frac{i}{2}(\vec{\omega}\,-i\vec{\nu})\cdot\vec{\sigma}} \,=\, \left(\Lambda_L^{-1}\right)^\dagger\,. \tag{2.26}$$

It can be shown that

$$\psi_L^\dagger\sigma^\mu\psi_L \tag{2.27}$$

where $\sigma^\mu \,=\, (1,\,\vec{\sigma})$, transforms as a vector. The same thing holds for

$$\chi_R^\dagger\overline{\sigma}^\mu\chi_R \tag{2.28}$$

where $\overline{\sigma}^\mu = (1,\,-\vec{\sigma})$.

In terms of two left-handed Weyl spinors $\psi,\ \chi$, we may introduce a four-component *Dirac spinor* as

$$\Psi \,=\, \begin{bmatrix} \psi \\ \\ i\sigma_2\chi^* \end{bmatrix}\,. \tag{2.29}$$

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[2]It can be shown that a complex 2×2 matrix

$$M \,=\, z_0 \,+\, \vec{z}\cdot\vec{\sigma},$$

subject to the condition $z_0^2 - \vec{z}\cdot\vec{z} = 1$ can always be written as an exponential $e^{i\vec{\xi}\cdot\vec{\sigma}}$ in terms of the complex vector

$$\vec{\xi} \,=\, -i\frac{\vec{z}}{\sqrt{z_0^2-1}}\left[\ln(z_0+\sqrt{z_0^2-1})\right]\,.$$

This can be further written as $e^{\frac{i}{2}(\vec{\omega}\,+\,i\vec{\nu})\cdot\vec{\sigma}}$.

[3]It can be shown that for the above introduced χ_R a Lorentz transformation Λ_R leaves $\chi^2 \,\equiv\, \chi_R\epsilon\chi_R$ invariant.

Since the left-handed and right-handed parts of a Dirac spinor transform independently under Lorentz transformations, the transformation matrix will have a block-diagonal form

$$\Psi'(x') = \begin{bmatrix} \psi'(x') \\ \\ i\sigma_2 \chi'^{*}(x') \end{bmatrix} = \begin{bmatrix} \Lambda_L \, \psi(x) \\ \\ \Lambda_R \, i\sigma_2 \chi^{*}(x) \end{bmatrix}$$

or

$$\Psi'(x') = \begin{pmatrix} e^{\frac{i}{2}(\vec{\omega}+i\vec{\nu})\cdot\vec{\sigma}} & 0 \\ \\ 0 & e^{\frac{i}{2}(\vec{\omega}-i\vec{\nu})\cdot\vec{\sigma}} \end{pmatrix} \Psi(x) \, . \tag{2.30}$$

Thus, Dirac spinors are reducible quantities under Lorentz transformations. The ultimate spinor building blocks are Weyl spinors.