Chapter 3

THE (CLASSICAL) ACTION PRINCIPLE

3.1 The Action Principle in Mechanics

Consider a system of N degrees of freedom described by a *Lagrangean* $L(q_j,\, \dot{q}_j)$ as a function of the generalized coordinates

$$q_1(t),\, q_2(t),\, \ldots,\, q_N(t)$$

and their velocities. The *Action* functional of the system between time t_i and t_f is defined as

$$\mathcal{S}[\underline{q};\, i_i,\, t_f] \equiv \int_{t_i}^{t_f} dt\, L(\underline{q},\, \dot{\underline{q}})\,. \tag{3.1}$$

The coordinates $\underline{q}(t)$ describe paths in *Configuration Space*. The actual physical trajectory of the system is one of these paths.

Next, consider functional variations of the coordinates $\delta q_j(t)$ that vanish at the end points, i.e.

$$\delta q_j(t_i) \,=\, \delta q_j(t_f) \,=\, 0\,. \tag{3.2}$$

There will be a corresponding variation of the Action $\delta\mathcal{S}$. The *Principle of Least Action* or *Hamilton's Principle* states that the actual physical trajectory of the system corresponds to a *minimum* of the Action

$$\Delta\mathcal{S} \,=\, 0\,. \tag{3.3}$$

This implies[1]

$$\int_{t_i}^{t_f} dt \sum_{i=1}^{N} \left\{ \frac{\partial L}{\partial q_j}\,\delta q_j \,+\, \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j \right\} =$$

$$\int_{t_i}^{t_f} dt \sum_{i=1}^{N} \left\{ \frac{\partial L}{\partial q_j}\,\delta q_j \,+\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\delta q_j\right) \,-\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j \right\}$$

$$= \int_{t_i}^{t_f} \sum_{j} \left\{ \frac{\partial L}{\partial q_j} \,-\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \right\} \delta q_j \,,$$

or, since the δq's are arbitrary,

$$\frac{\partial L}{\partial q_j} \,-\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \,=\, 0\,. \tag{3.4}$$

These are the *Euler-Lagrange Equations of Motion* of the system.

The *canonical momentum* p_i is defined as $p_i \,=\, \frac{\partial L}{\partial \dot{q}_i}$. In terms of it, we may introduce the description of the system in *Phase Space* $(q_i,\, p_i)$. The *Hamiltonian* is defined as

$$H(q,\, p) \,=\, \sum_i \dot{q}_i p_i \,-\, L\,. \tag{3.5}$$

Starting from the Lagrangean formalism and replacing the velocities with the momenta, the motion of the system can be formulated in terms of H and the *Hamilton Equations of Motion*

$$\dot{q}_i \,=\, \frac{\partial H}{\partial p_i},\ \dot{p}_i \,=\, -\frac{\partial H}{\partial q_i}\,. \tag{3.6}$$

A free particle of mass m is described by the Action

$$\mathcal{S} \,=\, -m\int ds \,=\, -m\int dt\sqrt{1-\left(\frac{d\vec{x}}{dt}\right)^2} \Longrightarrow L \,=\, -m\sqrt{1-(\dot{\vec{x}})^2}$$

From the canonical momentum of the particle we can arrive at the Hamiltonian

$$\vec{p} \,=\, \frac{m\dot{\vec{x}}}{\sqrt{1-(\dot{\vec{x}})^2}} \Longrightarrow \dot{\vec{x}} \,=\, \frac{\vec{p}}{\sqrt{p^2+m^2}} \Longrightarrow H \,=\, \dot{\vec{x}}\cdot\vec{p}-L \,=\, \sqrt{p^2+m^2}\,.$$

[1]Note that, since the variations are purely functional, and, therefore, instantaneous, the variation commutes with the time derivative, i.e. $\delta\dot{q} \,=\, \frac{d}{dt}(\delta q)$.

3.2 The Action Principle in Field Theory

The above procedure can be generalized and applied to continuous systems described by fields $\phi_a(x^\mu)$, where $x^\mu = (x^0, x^i) = (t, \vec{x})$. The index a stands both for spacetime indices, in case we have vector, spinor or tensor fields, and for internal indices corresponding to multiple fields. Expressing the Lagrangean in terms of a *Lagrange density* $\mathcal{L}(\phi, \partial_\mu\phi)$, we may write the Action in the form

$$\mathcal{S}[\phi] = \int_{t_i}^{t_f} dt\, L(\phi, \partial_\mu\phi) = \int_{t_i}^{t_f} dt \int d^3x\, \mathcal{L}(\phi, \partial_\mu\phi) \tag{3.7}$$

or

$$\mathcal{S}[\phi] = \int_{\sigma_i}^{\sigma_f} d^4x\, \mathcal{L}(\phi, \partial_\mu\phi), \tag{3.8}$$

where $\sigma_{i,f}$ are boundary space-like surfaces.

Introducing variations $\delta\phi_a(x)$ that vanish at the boundary

$$\delta\phi_a(\sigma_i) = \delta\phi_a(\sigma_f) = 0 \tag{3.9}$$

and demanding a minimum of the Action $\delta\mathcal{S} = 0$ for the physical system trajectory, we obtain

$$\int d^4x \sum_a \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a}\,\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\,\delta\phi_a\right) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\right)\delta\phi_a \right\}$$

$$= \int d^4x \sum_a \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\right)\right\}\delta\phi_a = 0$$

or

$$\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\right) = 0\,. \tag{3.10}$$

These are the *Euler-Lagrange Equations of Motion* for the fields ϕ_a.

The canonical formalism[2] can be set up as in the case of systems with finite degrees of freedom. The *canonical momenta* corresponding to the fields ϕ_a are defined as

$$\pi_a(x) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}_a}\,. \tag{3.11}$$

[2]The description of a system in terms of fields and momenta is referred to as *canonical formalism*.

The *Hamiltonian* of the system is

$$H \equiv \int d^3x \, \mathcal{H}(\phi, \pi) \tag{3.12}$$

in terms of the *Hamiltonian density*, defined as

$$\mathcal{H}(\phi_a, \pi_a) \equiv \dot{\phi}_a \, \pi_a - \mathcal{L} \, . \tag{3.13}$$

The evolution of the system of fields in the canonical formalism is described by *Hamilton's Equations*

$$\dot{\phi}_a = \frac{\partial \mathcal{H}}{\partial \pi_a}, \qquad \dot{\pi}_a = -\frac{\partial \mathcal{H}}{\partial \phi_a} \, . \tag{3.14}$$

An example of a relativistic continuous system is supplied by a real scalar field. The simplest Lagrange density that describes a free scalar field is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 \tag{3.15}$$

and leads to the *Klein-Gordon equation*

$$\left(\Box + m^2 \right) \phi(x) = 0 \, . \tag{3.16}$$

The general solution of the Klein-Gordon equation can be expanded in plane waves $e^{-ik\cdot x}$ as

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left(a(k) \, e^{-ik\cdot x} + a^*(k) \, e^{ik\cdot x} \right) , \tag{3.17}$$

where $\omega(k) = k^0 = \sqrt{\vec{k}^2 + m^2}$.

The canonical momentum corresponding to $\phi(x)$ is $\pi(x) = \dot{\phi}(x)$. The Hamiltonian is

$$H = \int d^3x \, \frac{1}{2} \left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2 \right) . \tag{3.18}$$