Chapter 4

FIELD QUANTIZATION

4.1 Canonical Commutation Relations

The passage from the classical description of a system of finite degrees of freedom $\{q_j(t),\, \dot{q}_j(t)\}$ to its quantum description goes through the canonical formalism $\{q_j(t),\, p_j(t)\}$ and consists in introducing operators $\{\hat{q}_j(t),\, \hat{p}_j(t)\}$ corresponding to the phase space variables. These operators are defined by their *equal time commutation relations*

$$
\begin{aligned}
[\, \hat{q}_j(t),\, \hat{p}_k(t)\,] &= i\,\delta_{jk} \\
[\, \hat{q}_j(t),\, \hat{q}_k(t)\,] &= [\, \hat{p}_j(t),\, \hat{p}_k(t)\,] = 0\,.
\end{aligned}
\tag{4.1}
$$

This procedure can also be followed in the case of a system of fields (*canonical quantization*). Starting point in the theory of quantized fields is to promote the fields $\phi_a(x)$ and their canonical momenta $\pi_a(x)$ into (Heisenberg) operators acting on the Hilbert space of quantum states. The next step is to impose on them *equal* *time commutation relations*[1]

$$
\begin{aligned}
[\,\phi_a(\vec{x},\,x_0),\,\phi_b(\vec{x}\,',\,x_0)\,] &= 0 \\
[\,\pi_a(\vec{x},\,x_0),\,\pi_b(\vec{x}\,',\,x_0)\,] &= 0 \\
[\,\phi_a(\vec{x},\,x_0),\,\pi_b(\vec{x}\,',\,x_0)\,] &= i\delta_{ab}\,\delta(\vec{x}-\vec{x}\,')\,.
\end{aligned}
\tag{4.2}
$$

For economy of notation we symbolize quantum fields in terms of the same symbols as classical fields, having dropped the hats denoting operators.

4.2 Lorentz Transformations for Quantum Fields

The requirement of Lorentz invariance of the theory is now carried over from the *classical fields* $\phi(x)$ and $\phi'(x')$ to the matrix elements of the *quantum fields* $\langle\psi_a|\phi(x)|\psi_b\rangle$ and $\langle\psi'_a|\phi(x')|\psi'_b\rangle$. Note the correspondence[2]

$$
\begin{aligned}
\phi(x) &\rightarrow \langle\psi_a|\phi(x)|\psi_b\rangle \\
\phi'(x') &\rightarrow \langle\psi'_a|\phi(x')|\psi'_b\rangle
\end{aligned}
\tag{4.3}
$$

For a scalar field, we conclude by correspondence that

$$
\phi'(x') = \phi(x) \rightarrow \langle\psi'_a|\phi(x')|\psi'_b\rangle = \langle\psi_a|\phi(x)|\psi_b\rangle\,. \tag{4.4}
$$

For a vector field, we conclude that

$$
A'^{\mu}(x') = \Lambda^{\mu}_{\ \nu}\,A^{\nu}(x) \rightarrow \langle\psi'_a|A^{\mu}(x')|\psi'_b\rangle = \Lambda^{\mu}_{\ \nu}\,\langle\psi_a|A^{\nu}(x)|\psi_b\rangle\,. \tag{4.5}
$$

These relations can be satisfied provided that the *"primed"* states are related to the original states through a *unitary transformation*

$$
|\psi'\rangle = U(\Lambda)|\psi\rangle\,. \tag{4.6}
$$

[1] The non-covariant choice of the surface $x'_0 = x_0$ can be replaced by the covariant notion of a space-like surface σ, i.e a three-dimensional surface whose normal η^{μ} is time-like ($\eta^2 = 1 > 0,\ \ \eta^0 > 0$).

[2] In the classical case for primed spacetime points we have primed fields, while in the quantum case for primed spacetime points we have primed states.

Then, the relation between the matrix elements gives

$$\phi(x') = U(\Lambda)\,\phi(x)\,U^{-1}(\Lambda)$$
$$A_\mu(x') = \Lambda^\mu_{\ \nu}\,U(\Lambda)\,A^\nu(x)\,U^{-1}(\Lambda) \tag{4.7}$$

or

$$U(\Lambda)\,\phi(x)\,U^{-1}(\Lambda) = \phi(x')$$
$$U(\Lambda)\,A^\mu(x)U^{-1}(\Lambda) = (\Lambda^{-1})_\nu^{\ \mu}\,A^\nu(x') \tag{4.8}$$

Since, all fields behave like scalar fields under Poincaré transformations, we also have, for a spacetime translation $x^\mu \to x^\mu + \alpha^\mu$,

$$\phi_b(x') = U(\alpha)\,\phi_b(x)\,U^{-1}(\alpha) \tag{4.9}$$

for any type of fields $\phi_b(x)$.

The unitary operators $U(\alpha)$ and $U(\Lambda)$ can always be written in terms of a set of Hermitean operators as

$$U(\alpha) = e^{i\alpha^\mu \hat{P}_\mu}, \qquad U(\Lambda) = e^{\frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}}\,. \tag{4.10}$$

In order to identify $\hat{P}_\mu$, let's substitute $U(\alpha)$ in (4.9) in the case of an infinitesimal translation. We have

$$\phi(x) + \alpha\cdot\partial\phi(x) = \phi(x) + i\alpha\cdot\left[\,\hat{P},\,\phi(x)\right]$$

or

$$\left[\,\hat{P}_\mu,\,\phi(x)\right] = -i\partial_\mu\phi(x) \tag{4.11}$$

which identifies $\hat{P}^\mu$ as the *momentum* operator of the system and the generator of spacetime translations. Note that the time component of this equation is just the Heisenberg equation for the field operator

$$\left[\,\hat{P}_0,\,\phi(x)\right] = -i\dot{\phi}(x)\,, \tag{4.12}$$

since $P_0 = H$ is the Hamiltonian of the system.

Doing the analogous thing for Lorentz transformation of a vector field, after some effort, we obtain

$$\left[\,\hat{J}^{\alpha\beta},\,A^\mu\right] = \left(\mathcal{J}^{\alpha\beta}\right)^\mu_{\ \ \nu}\,A^\nu \tag{4.13}$$

in terms of the generators $\mathcal{J}_{\mu\nu}$ introduced in (2.16). It can be verified in a staraightforward fashion that the ten generators $\hat{P}^\mu$, $\hat{J}^{\mu\nu}$ *satisfy exactly the same commutator Algebra, the Lorenz-Poincaré Algebra (1.17), (1.25), as the quantities* $\mathcal{P}^\mu$, $\mathcal{J}^{\mu\nu}$

$$
\begin{aligned}
[\hat{J}_{\mu\nu},\, \hat{J}_{\rho\sigma}] &= ig_{\nu\rho}\hat{J}_{\mu\sigma} - ig_{\mu\rho}\hat{J}_{\nu\sigma} - ig_{\nu\sigma}\hat{J}_{\mu\rho} + ig_{\mu\sigma}\hat{J}_{\nu\rho} \\
\left[\hat{J}_{\mu\nu}, \hat{P}_\rho\right] &= -ig_{\mu\rho}\hat{P}_\nu + ig_{\nu\rho}\hat{P}_\mu
\end{aligned}
\tag{4.14}
$$

Nevertheless, they are operators acting in the Hilbert space of quantum states in contrast to the latter, which are just matrices or differential operators introduced to express the action of the Lorentz-Poincaré Group on spacetime points and classical fields. The quantum generators $\hat{P}^\mu$, $\hat{J}^{\mu\nu}$ are expressible in terms of field operators and the canonical commutator relations are crucial in satisfying the Lorentz-Poincaré Algebra. The sets $\hat{P}^\mu$, $\hat{J}^{\mu\nu}$ and $\mathcal{P}^\mu$, $\mathcal{J}^{\mu\nu}$ are two *distinct representations* of the abstract generators of the Lorentz-Poincaré Group.