Chapter 5

SCALAR FIELDS

5.1 The Hermitean Free Scalar Field

Consider the Lagrange density of a Hermitean free scalar field leading to the Klein-Gordon equation of motion

$$\mathcal{L} \,=\, \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 \Longrightarrow \left(\Box + m^2\right)\phi \,=\, 0\,.$$

The canonical momentum and the corresponding Hamiltonian density are

$$\pi \,=\, \dot{\phi}, \quad \mathcal{H} \,=\, \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2\,. \tag{5.1}$$

Since $(\Box + m^2)e^{\pm ik\cdot x} = 0$ for $k_0 = \omega = \sqrt{\vec{k}^2 + m^2}$, the general solution of Klein-Gordon equation is

$$\phi(x) \,=\, \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left(a(k)\, e^{-ik\cdot x} \,+\, a^\dagger(k)\, e^{ik\cdot x} \right)\,, \tag{5.2}$$

where $a(k)$, $a^\dagger(k)$ are operator coefficients. The corresponding field momentum is

$$\pi(x) \,=\, -i\int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\omega}{2}} \left(a(k)\, e^{-ik\cdot x} \,-\, a^\dagger(k)\, e^{ik\cdot x} \right)\,. \tag{5.3}$$

Substituting in the canonical commutator $[\phi(x),\, \pi(x')]$ at equal times $x_0 = x'_0$, we obtain

$$[\,\phi(x),\, \pi(x')\,]|_{x_0 = x'_0} \,=\, -i\int\int \frac{d^3k\, d^3k'}{2(2\pi)^3}\sqrt{\frac{\omega'}{\omega}} \Big([\,a(k),\, a(k')]\, e^{-ik\cdot x - ik'\cdot x'}$$

$$- \left[a^\dagger(k),\, a^\dagger(k')\right] e^{ik\cdot x + ik'\cdot x'} - \left[a(k),\, a^\dagger(k')\right] e^{-ik\cdot x + ik'\cdot x'} + \left[a^\dagger(k),\, a(k')\right] e^{ik\cdot x - ik'\cdot x'} \Big) \,.$$

Assuming the following commutation relations on the a, $a^\dagger$'s

$$\begin{aligned} \left[a(k),\, a^\dagger(k')\right] \,&=\, \delta(\vec{k} - \vec{k}\,') \\ [a(k),\, a(k')] \,=\, 0, \quad \left[a^\dagger(k),\, a^\dagger(k')\right] \,&=\, 0 \end{aligned} \tag{5.4}$$

we obtain

$$[\,\phi(x),\, \pi(x')\,]|_{x_0 = x_0'} \,=\, i \int \int \frac{d^3k\, d^3k'}{2(2\pi)^3} \sqrt{\frac{\omega'}{\omega}} \left(\delta(\vec{k} - \vec{k}\,')\, e^{i\vec{k}\cdot(\vec{x} - \vec{x}\,')} + \delta(\vec{k} - \vec{k}\,')\, e^{-i\vec{k}\cdot(\vec{x} - \vec{x}\,')} \right)$$

$$= i \int \frac{d^3k}{2(2\pi)^3} \left(e^{i\vec{k}\cdot(\vec{x} - \vec{x}\,')} + e^{-i\vec{k}\cdot(\vec{x} - \vec{x}\,')} \right) \,=\, i \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x} - \vec{x}\,')}$$

and recover the canonical commutation relations

$$[\,\phi(\vec{x},\, x_0),\, \pi(\vec{x}\,',\, x_0)\,] \,=\, i\delta(\vec{x} - \vec{x}\,')\,.$$

Note that the commutation relations (5.4), assumed for the operators $a(k)$, $a^\dagger(k)$, are of the familiar harmonic oscillator type.

Substituting the general solution into the expression for the Hamiltonian

$$H \,=\, \frac{1}{2} \int d^3x \left(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2 \right),$$

after some calculation, we obtain

$$H \,=\, \frac{1}{2} \int d^3k\, \omega(k) \left(a^\dagger(k)a(k) + a(k)a^\dagger(k) \right) \,. \tag{5.5}$$

The Hamiltonian can also be written as

$$H \,=\, \int d^3k\, \omega(k)\, a^\dagger(k)a(k) \,+\, \mathcal{E}_0\,,$$

where $\mathcal{E}_0$ is the infinite constant

$$\mathcal{E}_0 \,=\, \frac{1}{2}\tilde{\delta}(0) \int d^3k\, \omega(k)\,.$$

$\tilde{\delta}(0)$ is the momentum-space delta function at zero momentum. In what follows we shall subtract this infinite constant adopting as Hamiltonian the so-called *normal ordered Hamiltonian*[1]

$$H = \int d^3k\,\omega(k)\,a^\dagger(k)a(k)\,. \tag{5.6}$$

The following commutation relations are true

$$\begin{aligned} [\,H,\,a(k)\,] &= -\omega(k)\,a(k) \\ \left[H,\,a^\dagger(k)\right] &= \omega(k)\,a^\dagger(k) \end{aligned} \tag{5.7}$$

It is obvious that the Hamiltonian (5.6) is *positive fefinite* and it has non-negative eigenvalues, namely, for $H|E\rangle = E|E\rangle$ we have $E = \langle E|H|E\rangle$ and

$$E = \int d^3k\,\omega(k)\langle E|a^\dagger(k)a(k)|E\rangle = \int d^3k\,\omega(k)\,\,||a(k)|E\rangle||^2 \geq 0\,. \tag{5.8}$$

Thus, the lowest energy eigenvalue is zero and if the eigenstate of zero energy is denoted by $|0\rangle$

$$H|0\rangle = 0 \Longrightarrow \int d^3k\,\omega(k)\,\,||a(k)|0\rangle||^2 = 0 \Longrightarrow a(k)|0\rangle = 0\,. \tag{5.9}$$

Thus, the *vacuum state* has to be annihilated by the operators $a(k)$.

Let us next consider the state $a^\dagger(k)|0\rangle$ and act with the Hamiltonian on it

$$Ha^\dagger(k)|0\rangle = \int d^3q\,\omega(q)\,a^\dagger(q)a(q)\,a^\dagger(k)|0\rangle =$$

$$\int d^3q\,\omega(q)\,a^\dagger(q)\left(a^\dagger(k)a(q) + \delta(\vec{q}-\vec{k})\right)|0\rangle = \omega(k)\,a^\dagger(k)|0\rangle\,.$$

[1] The operators $a(k)$ will be identified shortly as *"annihilation operators"*. A product of a's and $a^\dagger$'s is called *"normal ordered"* when it has all is annihilation operators to the right. For example

$$N(a(k_1)\,a^\dagger(k_2)\,a(k_3)) = a^\dagger(k_2)\,a(k_1)\,a(k_3)\,.$$

Thus, the state $a^\dagger(k)|0\rangle$ obbeys

$$H\left(a^\dagger(k)|0\rangle\right) = \omega(k)\left(a^\dagger(k)|0\rangle\right) . \tag{5.10}$$

This amounts to identifying this state as a state of one particle of momentum $\vec{k}$, energy $\omega(k) = \sqrt{\vec{k}^2 + m^2}$ and mass m. We may write

$$|k\rangle = N\, a^\dagger(k)|0\rangle \tag{5.11}$$

with N a normalization constant to be determined shortly. Thus, the operators $a(k)$ and $a^\dagger(k)$ act as *annihilation* and *creation* operators in exactly the same fashion as in the simple harmonic oscillator, the difference being in the existence of a triple infinity of independent oscillators labeled by the continuous parameters $\vec{k}$. The Hamiltonian is the direct sum of single oscillator Hamiltonians $\sum_k H_k = \sum_k \omega_k a_k^\dagger a_k$.

Further action by the creation operator on the vacuum state gives as multiparticle states like

$$\left(a^\dagger(k_1)\right)^{N_1}\left(a^\dagger(k_2)\right)^{N_2}|0\rangle = |N_1 k_1,\, N_2 k_2\rangle$$

which is a state of N_1 particles of momentum k_1 and N_2 particles of momentum k_2. Thanks to the commutativity of creation operators, these states are symmetric in the interchange of particles, i.e.

$$|k_1,\, k_2\rangle = |k_2,\, k_1\rangle$$

signifying the fact that the corresponding particles are *bosons*.

5.1.1 Normalization of the one-particle momentum eigenstates

The usual non-relativistic normalization $\langle\vec{p}|\vec{q}\rangle = \delta(\vec{p} - \vec{q})$ is not adequate since the delta function is not a covariant object. Note that, under a boost in the $\hat{z}$ direction, p_μ transforms as

$$\begin{pmatrix} p'_0 \\ p'_1 \\ p'_2 \\ p'_3 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \gamma(p_0+\beta p_3) \\ p_1 \\ p_2 \\ \gamma(p_3+\beta p_0) \end{pmatrix}$$

or

$$\begin{matrix} p'_3 = \gamma(p_3+\beta E) \\ E' = \gamma(E+\beta p_3) \end{matrix} \quad \Longrightarrow \frac{dp'_3}{dp_3} = \gamma\left(1 + \beta\frac{dE}{dp_3}\right) = \gamma\left(1 + \beta\frac{p_3}{E}\right) = \frac{E'}{E}\,.$$

Thus, we have

$$1 = \int d^3p'\,\delta(\vec{p}\,') = \int d^3p\,\delta(\vec{p}) \Longrightarrow \delta(\vec{p}) = \frac{E'}{E}\delta(\vec{p}\,')\,.$$

This singles out the following normalization[2]

$$|k\rangle = \sqrt{2\omega(k)}\,a^\dagger(k)\,|0\rangle \tag{5.12}$$

that gives

$$\langle k|k'\rangle = 2\omega(k)\,\delta(\vec{k} - \vec{k}\,') \tag{5.13}$$

which is Lorentz invariant.

The completeness of these states gives

$$1_{1p} = \int \frac{d^3k}{2\omega(k)}\,|k\rangle\,\langle k|\,. \tag{5.14}$$

Note that the combination $\frac{d^3p}{2E}$ being Lorentz invariant, enables us to write

$$\int \frac{d^3p}{2E}\,\ldots = \int d^4p\,\delta(p^2-m^2)\,\Theta(p_0)\,\ldots\,. \tag{5.15}$$

[2]The factor 2, not necessary for covariance, is just a factor of convenience that simplifies certain expressions.

5.1.2 Physical Content of the State $\phi(\mathbf{x})|\mathbf{0}\rangle$

Acting on the vacuum with the field operator, we have

$$\begin{aligned}\phi(x)|0\rangle &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}}\, e^{i\omega t}\, e^{-i\vec{k}\cdot\vec{x}}\, a^\dagger(k)|0\rangle \\ &= \int \frac{d^3k}{(2\pi)^{3/2}2\omega}\, e^{i\omega t}\, e^{-i\vec{k}\cdot\vec{x}}\, |k\rangle\,.\end{aligned}$$

This shows that the state $\phi(x)|0\rangle$ is a linear combination of momentum eigenstates. Actually, this superposition would coincide with the corresponding non-relativistic expression if the denominator were absent, since the amplitude to find the particle in a state of momentum $\vec{k}$ is just $\frac{e^{-i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}$. Thus, we adopt the same interpretation here as well, namely, *that $\phi(x)|0\rangle$ is a state of a particle at the point $\vec{x}$, independently of its momentum.* In addition, since

$$\langle 0|\phi(x)|p\rangle = \int \frac{d^3k}{(2\pi)^{3/2}2\omega}\, e^{-i\omega t}\, e^{i\vec{k}\cdot\vec{x}}\, \langle k|p\rangle = \frac{e^{i\vec{p}\cdot\vec{x}}}{(2\pi)^{3/2}} e^{-iEt},$$

it is clear that $\langle 0|\phi(x)|p\rangle$ should be interpreted as the wave function of a particle in a state of definite momentum $|p\rangle$.

5.1.3 Lorentz Transformations on Creation/Annihilation Operators

A Lorentz transformation

$$x'^\mu = \Lambda^\mu_{\ \nu}\, x^\nu, \quad k'^\mu = \Lambda^\mu_{\ \nu}\, k^\nu \tag{5.16}$$

acts on the momentum eigenstates as a unitary operator $\mathbf{U}(\Lambda)$

$$|k'\rangle = U(\Lambda)|k\rangle\,. \tag{5.17}$$

This implies

$$\sqrt{2\omega(k')}\, a^\dagger(k')|0\rangle = \sqrt{2\omega(k)}\, U(\Lambda)\, a^\dagger(k)|0\rangle$$

or, since the vacuum is Lorentz-invariant and $U(\Lambda)|0\rangle = |0\rangle$,

$$\sqrt{\omega(k')}a^\dagger(k') = \sqrt{\omega(k)}U(\Lambda)a^\dagger(k)U^{-1}(\Lambda)\,. \tag{5.18}$$

Let's see if this is consistent with the relation $\phi(x') = U(\Lambda)\,\phi(x)\,U^{-1}(\Lambda)$ considered earlier. This relation amounts to

$$\int \frac{d^3k'}{(2\pi)^{3/2}\sqrt{2\omega(k')}}\left(\mathbf{a}^\dagger(k')e^{-ik'\cdot x'} + h.c.\right) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega(k)}}\left(U(\Lambda)a^\dagger(k)U^{-1}(\Lambda)e^{-ik\cdot x} + h.c.\right)$$

or

$$\int \frac{d^3k}{(2\pi)^{3/2}2\omega(k)}\sqrt{2\omega(k')}\left(\mathbf{a}^\dagger(k')e^{-ik\cdot x} + h.c.\right) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega(k)}}\left(U(\Lambda)a^\dagger(k)U^{-1}(\Lambda)e^{-ik\cdot x} + h.c.\right)$$

from which (5.18) is read off.

A spacetime translation

$$x'^\mu = x^\mu + \alpha^\mu \qquad (5.19)$$

acts on the field operator through the unitary operator $U(\alpha) = e^{i\alpha^\mu P_\mu}$ as

$$\phi(x') = U(\alpha)\,\phi(x)\,U^{-1}(\alpha)\,. \qquad (5.20)$$

From an infinitesimal translation, we get

$$[\,P_\mu,\,\phi(x)\,] = -i\partial_\mu\phi(x)\,. \qquad (5.21)$$

In the time direction this coincides with Heisenberg's equation

$$[\,P_0,\,\phi(x)\,] = -i\dot{\phi}(x)\,. \qquad (5.22)$$

Let's see if this relation is satisfied by the Hamiltonian $P_0 = H$ that we have derived. We have

$$\left[\int d^3q\,\omega(q)a^\dagger(q)a(q),\,\int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega(k)}}\left(a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}\right)\right] =$$

$$\int\int \frac{d^3qd^3k}{(2\pi)^{3/2}\sqrt{2\omega(k)}}\omega(q)\left(-\delta(\vec{q}-\vec{k})a(q)e^{-ik\cdot x} + \delta(\vec{q}-\vec{k})\mathbf{a}^\dagger(q)e^{ik\cdot x}\right)$$

$$= \int \frac{d^3k\omega(k)}{(2\pi)^{3/2}\sqrt{2\omega(k)}}\left(-a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}\right) = -i\dot{\phi}(x)\,.$$

Equivalently, we may use the expression of the Hamiltonian in terms of the field and its momentum and employ the canonical commutation relations:

$$[\,H,\,\phi(x)\,] = \frac{1}{2}\int d^3x'\left[\left(\pi^2(\vec{x}\,',\,x_0) + (\vec{\nabla}'\phi(\vec{x}\,',\,x_0))^2 + m^2\phi^2(\vec{x}\,',\,x_0)\right),\,\phi(\vec{x},\,x_0)\right]$$

$$= \frac{1}{2}\int d^3x' \Big[\pi^2(\vec{x}\,', x_0), \phi(\vec{x}, x_0) \Big] =$$

$$\frac{1}{2}\int d^3x' \; (\, \pi(\vec{x}\,', x_0) \,[\, \pi(\vec{x}\,', x_0), \phi(\vec{x}, x_0) \,] \; + \; [\, \pi(\vec{x}\,', x_0), \phi(\vec{x}, x_0) \,] \, \pi(\vec{x}\,', x_0) \,)$$

$$= \int d^3x' \, \pi(\vec{x}\,', x_0) \, (-i) \delta(\vec{x}\,' - \vec{x}) \; = \; -i\, \pi(x) \; = \; -i\dot{\phi}(x) \, .$$