Chapter 6

PROPAGATION AND CAUSALITY

6.1 A parenthesis: Propagation in Non-Relativistic QM

The amplitude for a free particle that propagates from $\vec{x}_0$ to a point $\vec{x}$ in time t is

$$U(t) \;=\; \langle \vec{x}|e^{-iHt}|\vec{x}_0\rangle \;=\; \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot(\vec{x}-\vec{x}_0)}\, e^{-it\frac{p^2}{2m}} \;=\; \left(\frac{m}{2\pi i t}\right)^{3/2} e^{\frac{im}{2t}(\vec{x}-\vec{x}_0)^2}$$

As could be expected, this is (wrongly) non-zero at $|\vec{x}-\vec{x}_0| > t$, i.e. outside of the light-cone. The situation might improve if we use the relativistic expression for the Hamiltonian $H = \sqrt{p^2+m^2}$. Then, we have

$$U(t) \;=\; \langle \vec{x}|e^{-iHt}|\vec{x}_0\rangle \;=\; \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot(\vec{x}-\vec{x}_0)}\, e^{-it\sqrt{p^2+m^2}}$$

$$= \frac{i}{4\pi^2 r}\int_{-\infty}^{+\infty} dp\, p\, e^{-ipr-it\sqrt{p^2+m^2}} \;=\; -\frac{1}{4\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{+\infty} dp\, e^{-ipr-it\sqrt{p^2+m^2}}\,.$$

The last integral can be evaluated with the saddle point method. Outside of the light-cone ($t^2 - |\vec{x} - \vec{x}_0|^2 = t^2 - r^2 < 0$), we have[1]

$$U(t) \approx -\frac{1}{4\pi^2 r}\frac{\partial}{\partial r}\left\{ e^{-m\sqrt{r^2-t^2}}\sqrt{\frac{2\pi m t^2}{(r^2-t^2)^{3/2}}}\right\} = e^{-m\sqrt{r^2-t^2}}\left(\ldots\right) .$$

Although, this amplitude is small, it is a non-zero amplitude to find the particle outside the light-cone.

6.2 The Real Thing: Propagation in Relativistic QFT

With the previously explained interpretation of $\phi(x)|0\rangle$ as the state of a particle at x, it is evident that the Lorentz invariant quantity

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \ldots = \int \frac{d^3p}{(2\pi)^3}\frac{e^{-ip\cdot(x-y)}}{2E}$$

must be interpreted as the amplitude to go from y to x. If we evaluate it outside the light cone ($(x-y)^2 < 0$)[2], we obtain $D(r) \approx e^{-mr}/r^2$, i.e. it decreases rapidly but *it is non-zero.* Actually, the

[1] For an integral of this form the saddle point method gives

$$\int_{-\infty}^{+\infty} dp\, e^{-f(p)} \approx e^{-f(p_0)}\sqrt{\frac{2\pi}{f''(p_0)}}$$

with $f'(p_0) = 0$. In our case we have $f(p) = -ipr + it\sqrt{p^2+m^2}$ and $f'(p_0) = 0$ gives

$$p_0 = \frac{imr}{\sqrt{r^2-t^2}}, \quad f(p_0) = m\sqrt{r^2-t^2}, \quad f''(p_0) = \frac{(r^2-t^2)^{3/2}}{mt^2}$$

[2] Taking $x^0 = y^0 = 0$ and $\vec{x} - \vec{y} = \vec{r}$, we have

$$D(r) = \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot\vec{r}}}{2E} = -\frac{i}{8\pi^2 r}\int_{-\infty}^{+\infty} dp\, \frac{p}{\sqrt{p^2+m^2}}e^{ipr} .$$

Deforming the contour, so that it goes around the imaginary axis $p \in [im, i\infty]$, we get

$$\int_{-\infty}^{+\infty} dp\, \frac{p}{\sqrt{p^2+m^2}}e^{ipr} = -\int_{i\infty}^{im} d(i\xi)\frac{i\xi}{\sqrt{(i\xi)^2+m^2}}e^{i(i\xi)r} + \int_{im}^{i\infty} d(i\xi)\frac{i\xi}{\sqrt{(i\xi)^2+m^2}}e^{i(i\xi)r}$$

right question to ask is not whether particles can propagate outside the light cone but *whether a measurement performed at a point* x *can affect another measurement performed at a point* y*, separated from it with a spacelike distance* $(x-y)^2 < 0$. If the measurement has to do with the particle created by ϕ, the above statement can be translated into asking whether

$$[\,\phi(x),\,\phi(y)\,] \,=\, 0, \quad (x-y)^2 < 0 \tag{6.1}$$

This property is called *microcausality*. This commutator is

$$[\,\phi(x),\,\phi(y)\,] \,=\, \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E}\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) \,=\, D(x-y) - D(y-x)\,.$$

Nevertheless, this is zero, because, for spacelike $x-y$, we can perform a continuous transformation and take it to $-(x-y)$. This is something that cannot be done for a timelike vector, since the points of the upper light-cone cannot be Lorentz-transformed to points of the lower light-cone.

Interpretation: The amplitude $\langle 0|\phi(x)\phi(y)|0\rangle$ corresponds to a particle created at y and destroyed at x. The other term in the commutator, namely $\langle 0|\phi(y)\phi(x)|0\rangle$, corresponds to an antiparticle created at x and destroyed at y. It just happens that the two amplitudes exactly cancel and preserve causality. This interpretation becomes apparent in the case of the charged scalar field, where we have the amplitudes $\langle 0|\phi(x)\phi^\dagger(y)|0\rangle$, corresponding to a charged particle created at y and destroyed at x, and $\langle 0|\phi^\dagger(y)\phi(x)|0\rangle$, corresponding to particle *of opposite charge* created at x and destroyed at y.

6.3 The Feynman Propagator

Consider the Green's functions equation corresponding to the Klein-Gordon operator

$$\left(\partial^2 + m^2\right) D(x-y) \,=\, -\delta(x-y)\,. \tag{6.2}$$

$$= 2i\int_m^\infty d\xi \frac{\xi}{\sqrt{\xi^2-m^2}} r^{-\xi r} \approx|_{r\to\infty} 2i\int_m^\infty d\xi\, e^{-\xi r} \,=\, 2i\,\frac{e^{-mr}}{r}\,.$$

Thus, for $r >> m$, we have $D(r) \approx \frac{e^{-mr}}{(2\pi r)^2}$.

Introducing the Fourier transform

$$D(x-y) \;=\; \int \frac{d^4k}{(2\pi)^2}\,\tilde{D}(k)\,e^{-ik\cdot(x-y)} \tag{6.3}$$

we obtain

$$(k^2\;-m^2)\,\tilde{D}(k) \;=\; \frac{1}{(2\pi)^2}$$

or

$$D(x-y) \;=\; \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2\;-m^2}\,. \tag{6.4}$$

As it stands this integral is undefined, since it has poles along the integration path, namely at $\pm\omega \;=\; \pm\sqrt{\vec{k}^2+m^2}$, and we need a specific prescription how to go around the poles. To go around the poles we use *Feynman's prescription*, corresponding to including the ω pole for $x_0 > y_0$ (lower contour) and the $-\omega$ pole for $x_0 < y_0$ (upper contour). The evaluation goes as follows:

$$\int \frac{d^3k}{(2\pi)^4}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\int dk_0\,\frac{e^{-ik_0(x_0-y_0)}}{k_0^2-(\vec{k}^2+m^2)} \;=\; \int \frac{d^3k}{(2\pi)^4}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\int dk_0\,\frac{e^{-ik_0(x_0-y_0)}}{k_0^2-\omega^2} \;=$$

$$\int \frac{d^3k}{(2\pi)^4}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left\{\Theta(x_0-y_0)\left(\frac{2\pi i}{-2\omega}\right)e^{-i\omega(x_0-y_0)} \;+\; \Theta(y_0-x_0)\left(\frac{-2\pi i}{2\omega}\right)e^{i\omega(x_0-y_0)}\right\}$$

$$=\; -i\Theta(x_0-y_0)\int \frac{d^3k}{(2\pi)^3 2\omega}e^{-ik\cdot(x-y)} \;-i\Theta(y_0-x_0)\int \frac{d^3k}{(2\pi)^3 2\omega}e^{ik\cdot(x-y)}\,,$$

with $\omega \;=\; \sqrt{\vec{k}^2+m^2}$. This corresponds to giving a negative imaginary part to the pole $\omega \to \omega - i\epsilon$ and a positive imaginary part to the pole $-\omega \to\; -\omega + i\epsilon$. This is equivalent to the shift $m^2 \;\to\; m^2\;-i\epsilon$.

$$D_F(x-y) \;\equiv\; \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2-m^2+i\epsilon}\,. \tag{6.5}$$

Next, we may consider the scalar field $\phi(x)$ and using its expansion in terms of creation-annihilation operators, we can prove that

$$\langle 0|\phi(x)\phi(y)|0\rangle \;=\; \int \frac{d^3k}{(2\pi)^3 2\omega}e^{-ik\cdot(x-y)} \tag{6.6}$$

and, therefore,

$$D_F(x-y) \,=\, -i\Theta(x_0-y_0)\langle 0|\phi(x)\phi(y)|0\rangle \,-i\Theta(y_0-x_0)\langle 0|\phi(y)\phi(x)|0\rangle\,. \tag{6.7}$$

Introducing the *Time Ordered Product*

$$T\phi(x)\phi(y) \,=\, \Theta(x_0-y_0)\phi(x)\phi(y) \,+\, \Theta(y_0-x_0)\phi(y)\phi(x) \tag{6.8}$$

we can write

$$D_F(x-y) \,=\, -i\langle 0|\, T\phi(x)\phi(y)\,|0\rangle\,. \tag{6.9}$$

For the hermitian scalar field, a particle coincides with its own antiparticle. This is not the case for a complex scalar field. If we had a complex field

$$\phi(x) \,=\, \sum \left(a\, e^{-ik\cdot x} \,+\, b^\dagger e^{ik\cdot x}\right) \tag{6.10}$$

the expression of the Feynman propagator as a time-ordered product would be instead

$$D_F(x-y) \,=\, -i\langle 0|T\phi(x)\phi^\dagger(y)|0\rangle\,. \tag{6.11}$$

Thus, $\phi(x)\phi^\dagger(y)$ and $\phi^\dagger(y)\phi(x)$ would show up, with the following interpretation:

$\langle 0|\phi(x)\phi^\dagger(y)|0\rangle \Longrightarrow aa^\dagger\, e^{-ik_0(x_0-y_0)}$ a particle is created at y and it propagates to x, where it is destroyed (at a later time $x^0 > y^0$)

and

$\langle 0|\phi^\dagger(y)\phi(x)|0\rangle \Longrightarrow bb^\dagger\, e^{-ik_0(y_0-x_0)}$ an antiparticle is created at x and it propagates to y where it is destroyed (at a later time $y^0 > x^0$).

As a result, either a particle goes from y to x or an antiparticle goes from x to y but particles or antiparticles always propagate with a positive frequency (energy).

6.3.1 Analytic Expressions for the Feynman and other Propagators

We first introduce the odd and even solutions of the Klein-Gordon equation

$$\Delta(x-x') \,=\, -i\int \frac{d^3k}{(2\pi)^3 2\omega}\left(e^{-ik\cdot(x-x')} \,-\, e^{ik\cdot(x-x')}\right) \tag{6.12}$$

$$\Delta_1(x-x') \,=\, \int \frac{d^3k}{(2\pi)^3 2\omega} \left(e^{-ik\cdot(x-x')} \,+\, e^{ik\cdot(x-x')} \right) \tag{6.13}$$

They are both solutions to the Klein-Gordon equation

$$\left(\partial^2 \,+\, m^2\right)\Delta(x-x') \,=\, 0\,, \qquad \left(\partial^2 \,+\, m^2\right)\Delta_1(x-x') \,=\, 0\,. \tag{6.14}$$

Note that

$$\Delta(x'-x) \,=\, -\Delta(x-x'), \quad \Delta_1(x'-x) \,=\, +\Delta_1(x-x')\,.$$

The following can be shown[3]:

$$D_F(x-x') \,=\, \frac{1}{2}\left\{\, 2sign(x_0-x_0')\Delta(x-x') \,-\, i\Delta_1(x-x') \,\right\}\,.$$

The odd function $\Delta(x-x')$ vanishes outside the light-cone, i.e.

$$\Delta(x-x') \,=\, 0 \qquad \forall (x-x')^2 < 0$$

and it is singular on the light-cone

$$\left.\frac{\partial\Delta(x)}{\partial x_0}\right|_{x_0=0} \,=\, \delta(\vec{x})\,.$$

It can be calculated, integrating its defining expression, to be

$$\Delta(x) \,=\, \frac{1}{4\pi r}\frac{\partial}{\partial r}\begin{cases} J_0(m\sqrt{t^2-r^2}) & t>r \\ 0 & -r<t<r \\ -J_0(m\sqrt{t^2-r^2}) & t<-r \end{cases} \tag{6.15}$$

where J_0 is the Bessel function of the first kind.

The even function $\Delta_1(x-x')$ does not vanish outside the light-cone but decreases exponentially. It is

$$\Delta_1(x) \,=\, \frac{1}{4\pi r}\frac{\partial}{\partial r}\begin{cases} Y_0(m\sqrt{t^2-r^2}) & t^2>r^2 \\ -\frac{2}{\pi}K_0(m\sqrt{r^2-t^2}) & t^2<r^2 \end{cases} \tag{6.16}$$

[3]Note that $sign(x) = 2\Theta(x) - 1$.

where K_0 is the Bessel function of the second kind and Y_0 is the modified Bessel function.

In the massless $m = 0$ case, we have

$$\begin{aligned} \Delta(x) \; &= \; \tfrac{1}{4\pi r} \left[\, \delta(r + t) \; - \delta(r - t) \, \right] \\ \Delta_1(x) \; &= \; \tfrac{1}{4\pi^2 r} \left[\, P \tfrac{1}{r+t} \; + \; P \tfrac{1}{r-t} \, \right] \end{aligned} \tag{6.17}$$

where $P\frac{1}{r \pm t}$ stands for the (Cauchy) principal value.