Chapter 7

SYMMETRIES AND CONSERVATION LAWS (NOETHER'S THEOREM)

7.1 Noether's Theorem in Mechanics

For every symmetry of the Action there is a conservation law, expressed through coressponding conserved currents, characteristic of the symmetry at hand. This is the so-called **Noether's Theorem**. Its proof for a system with a finite number of degrees of freedom goes as follows:

Consider the *unconstrained variations* δq_j, $\delta_0 q_j$ and δt, defined as

$$\delta q_i = q_i'(t') - q_i(t), \quad \delta_0 q_i = q_i'(t) - q_i(t), \quad \delta t = t' - t\,. \tag{7.1}$$

Note that

$$\delta q_i(t) = \delta_0 q_i(t) + \delta t\, \dot{q}_i(t) \tag{7.2}$$

and

$$\frac{d}{dt}\delta_0 q_i(t) = \delta_0 \dot{q}_i(t)\,. \tag{7.3}$$

In terms of these variations we have

$$\delta\mathcal{S} \,=\, \int \left\{\, \delta dt\, L \,+\, dt\,\delta L \,\right\} \,=\, \int dt \left\{ \left(\frac{dt'}{dt} - 1 \right) L \,+\, \delta L \right\} \approx \int dt \left\{ \frac{d\delta t}{dt}\, L \,+\, \delta L \right\}.$$

On the other hand

$$\delta L \,=\, \delta t\, \dot{L} \,+\, \delta_0 L \,=\, \delta t\, \dot{L} \,+\, \frac{\partial L}{\partial q_i}\delta_0 q_i \,+\, \frac{\partial L}{\partial \dot{q}_i}\delta_0 \dot{q}_i \,.$$

Note that summation over all the i's is implied. Enforcing the equations of motion, this becomes

$$\begin{aligned} \delta L \,&=\, \delta t\, \dot{L} \,+\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta_0 q_i \,+\, \frac{\partial L}{\partial \dot{q}_i}\delta_0 \dot{q}_i \\ &=\, \delta t\, \dot{L} \,+\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta_0 q_i\right) . \end{aligned}$$

Thus, the variation of the Action is

$$\delta\mathcal{S} \,=\, \int dt \left\{ \frac{d}{dt}\left(\delta t\right)\, L \,+\, \delta t\, \dot{L} \,+\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta_0 q_i\right) \right\}$$

$$= \int dt\, \frac{d}{dt}\left(\delta t\, L \,+\, \frac{\partial L}{\partial \dot{q}_i}\delta_0 q_i\right) \,=\, \int dt\, \frac{d}{dt}\left(\left(L \,-\, \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\delta t \,+\, \frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) .$$

Invariance of the Action under the set of variations $\delta q_i, \delta t$ implies that the quantity

$$\mathcal{G} \,=\, \left(L \,-\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\delta t \,+\, \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta q_i \tag{7.4}$$

is conserved, i.e.

$$\delta\mathcal{S} \,=\, 0 \Longrightarrow \frac{d\mathcal{G}}{dt} \,=\, 0\,. \tag{7.5}$$

This is the so-called *Noether's Theorem.*

As an example consider invariance under time translations $\delta t = \epsilon$. The conserved quantity is

$$\mathcal{G} \,=\, \left(L \,-\, \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\epsilon\,,$$

which is equivalent to the conservation of energy

$$H = \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = p_i\dot{q}_i - L\,. \tag{7.6}$$

Similarly, invariance under the rotations $\delta x_i = \epsilon_{ijk}\varepsilon_j x_k$ gives the conserved quantity

$$\frac{\partial L}{\partial \dot{x}_i}\epsilon_{ijk}\varepsilon_j x_k = \vec{\varepsilon}\cdot(\vec{x}\times\vec{p}) \tag{7.7}$$

which is the standard angular momentum.

7.2 Noether's Theorem in Field Theory

The proof of Noether's theorem in Field Theory goes as follows:

Consider the variations

$$\delta\phi_a(x) = \phi'_a(x') - \phi_a(x),\ \delta_0\phi_a(x) = \phi'_a(x) - \phi_A(x), \quad \delta x^\mu = x'^\mu - x^\mu\,. \tag{7.8}$$

Note that

$$\delta_0\partial_\mu\{\ldots\} = \partial_\mu\delta_0\{\ldots\} \tag{7.9}$$

and

$$\delta\{\ldots\} = \delta_0\{\ldots\} + \delta x^\mu\,\partial_\mu\{\ldots\}\,. \tag{7.10}$$

The resulting variation in the Action is[1]

$$\delta\mathcal{S} = \int\Big\{\delta(d^4x)\mathcal{L} + d^4x\,\delta\mathcal{L}\Big\} \approx \int d^4x\,\{(\partial_\mu\delta x^\mu)\,\mathcal{L} + \delta x^\mu\,\partial_\mu\mathcal{L} + \delta_0\mathcal{L}\}$$

$$= \int d^4x\left\{(\partial_\mu\delta x^\mu)\,\mathcal{L} + \delta x^\mu\,\partial_\mu\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\phi_a}\delta_0\phi_a + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\delta_0\partial_\mu\phi_a\right\}$$

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[1]

$$\delta d^4x = d^4x' - d^4x = d^4x\left(Det(\partial_\mu x'^\nu) - 1\right) = d^4x\left(e^{Tr\ln(\partial_\mu x'^\nu)} - 1\right)$$

$$= d^4x\left(e^{Tr\ln(\delta^\nu_\mu + \partial_\mu\delta x^\nu)} - 1\right) \approx d^4x\left(e^{Tr(\partial_\mu\delta x^\nu)} - 1\right) \approx d^4x\,Tr(\partial_\mu\delta x^\nu) = d^4x\,\partial_\mu\delta x^\mu\,.$$

$$= \int d^4x \left\{ \partial_\mu \left(\delta x^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \delta_0 \phi_a \right) + \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \right) \right) \delta_0 \phi_a \right\}$$

Enforcing the equations of motion, we get

$$\delta \mathcal{S} = \int d^4x \, \partial_\mu \left(\delta x^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \delta_0 \phi_a \right) =$$

$$\int d^4x \, \partial_\mu \left(\delta x^\nu \left(\mathcal{L} g^\mu_\nu - \partial_\nu \phi_a \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \right) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \delta \phi_a \right) = \int d^4x \, \partial_\mu \mathcal{J}^\mu \, .$$

Thus, if the Action is invariant

$$\delta \mathcal{S} = 0 \Longrightarrow \partial_\mu \mathcal{J}^\mu = 0 \tag{7.11}$$

if the current $\mathcal{J}^\mu$ is conserved. This current is called *Noether Current*

$$\mathcal{J}^\mu = \delta x^\nu \left(\mathcal{L} \, g^\mu_\nu - \sum_a (\partial_\nu \phi_a) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \right) + \sum_a \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \delta \phi_a \, . \tag{7.12}$$

The *charges* corresponding to the conserved current $\mathcal{J}^\mu$ are constants of the motion, i.e.

$$\mathcal{Q} \equiv \int d^3x \, \mathcal{J}_0 \Longrightarrow \frac{d\mathcal{Q}}{dt} = \int d^3x \, \partial_0 \mathcal{J} = \int d^3x \, \partial_\mu \mathcal{J}^\mu - \int d^3x \, \vec{\nabla} \cdot \vec{\mathcal{J}}$$

$$= - \int d^3x \, \vec{\nabla} \cdot \vec{\mathcal{J}} = - \oint d\vec{S} \cdot \vec{\mathcal{J}} \to 0 \, .$$

7.3 The Energy-Momentum Tensor

Next, let us consider the case $\delta \phi_a = 0$, while δx^μ corresponds to a *Poincaré transformation*

$$\delta x^\mu = \epsilon^\mu \Longrightarrow \mathcal{J}^\mu = -\epsilon^\nu \, T^\mu{}_\nu \, , \tag{7.13}$$

where

$$T^\mu{}_\nu = -\mathcal{L} \, g^\mu_\nu + \sum_a (\partial_\nu \phi_a) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \, . \tag{7.14}$$

Since ϵ_μ is a constant vector, $T^\mu_{\ \nu}$, which is called the *energy-momentum tensor*[2], is conserved

$$\partial_\mu T^\mu_{\ \nu} = 0 \ . \tag{7.15}$$

Note that

$$T_{00} = -\mathcal{L} + \sum_a \dot{\phi}_a \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} = \sum_a \dot{\phi}_a \, \pi_a - \mathcal{L} = \mathcal{H} \, , \tag{7.16}$$

which is the Hamiltonian density of the system, and

$$T^i_0 = \sum_a \dot{\phi}_a \frac{\partial \mathcal{L}}{\partial \partial_i \phi_a} \, , \tag{7.17}$$

which is the field momentum density. The integrals

$$H = \int d^3x \, T_{00} \qquad P^i = \int d^3x \, T^i_0$$

or

$$\mathcal{P}^\mu = \int d^3x \, T^\mu_0 \tag{7.18}$$

correspond to the associated (conserved) charge, i.e. the four-momentum of the system.

The energy-momentum tensor for a Hermitean *scalar field* $\phi(x)$ with a Lagrangean of the form

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, g^{\mu\nu} \, \partial_\nu \phi - V(\phi)$$

is automatically symmetric. It comes out as

$$T_{\mu\nu} = -\mathcal{L} \, g_{\mu\nu} + \partial_\mu \phi \, \partial_\nu \phi \, . \tag{7.19}$$

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[2]It can be shown that the above energy-momentum tensor can also be obtained by considering general spacetime metric variations $\delta g_{\mu\nu}$ of the Action

$$\mathcal{S} = \int d^4x \sqrt{-g} \, \mathcal{L} \, ,$$

where $g = Det(g_{\mu\nu})$, as

$$\int d^4x \sqrt{-g} \, T^{\mu\nu} \, \delta g_{\mu\nu} = -\delta \mathcal{S} \, .$$

For a general field (vector, spinor,. . .) the canonical energy-momentum tensor, defined as above, is not necessarily symmetric. Nevertheless, a symmetric energy-momentum tensor can always be defined by adding a suitable conserved function to it.