Chapter 7

SYMMETRIES AND CONSERVATION LAWS (NOETHER’S THEOREM)

7.1 Noether’s Theorem in Mechanics

For every symmetry of the Action there is a conservation law, expressed through corresponding conserved currents, characteristic of the symmetry at hand. This is the so-called Noether’s Theorem. Its proof for a system with a finite number of degrees of freedom goes as follows:

Consider the unconstrained variations \( \delta q_i, \delta_0 q_i \) and \( \delta t \), defined as

\[
\delta q_i = q_i'(t') - q_i(t), \quad \delta_0 q_i = q_i'(t) - q_i(t), \quad \delta t = t' - t .
\]  

(7.1)

Note that

\[
\delta q_i(t) = \delta_0 q_i(t) + \delta t \dot{q}_i(t) \]  

(7.2)

and

\[
\frac{d}{dt} \delta_0 q_i(t) = \delta_0 \dot{q}_i(t) .
\]  

(7.3)

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In terms of these variations we have

$$\delta S = \int \left\{ \delta t dt L + dt \delta L \right\} = \int dt \left\{ \left( \frac{dt'}{dt} - 1 \right) L + \delta L \right\} \approx \int dt \left\{ \frac{d\delta t}{dt} L + \delta L \right\}. $$

On the other hand

$$\delta L = \delta t \dot{L} + \delta_0 L = \delta t \dot{L} + \frac{\partial L}{\partial q_i} \delta_0 q_i + \frac{\partial L}{\partial \dot{q}_i} \delta_0 \dot{q}_i. $$

Note that summation over all the $i$'s is implied. Enforcing the equations of motion, this becomes

$$\delta L = \delta t \dot{L} + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \delta_0 q_i + \frac{\partial L}{\partial \dot{q}_i} \delta_0 \dot{q}_i. $$

Thus, the variation of the Action is

$$\delta S = \int dt \left\{ \frac{d}{dt} (\delta t) L + \delta t \dot{L} + \frac{d}{dt} \left( \frac{\partial L}{\partial q_i} \right) \delta_0 q_i \right\}$$

$$= \int dt \frac{d}{dt} \left( \delta t L + \frac{\partial L}{\partial \dot{q}_i} \delta_0 q_i \right) = \int dt \frac{d}{dt} \left( L - \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \delta t + \frac{\partial L}{\partial \dot{q}_i} \delta q_i. $$

Invariance of the Action under the set of variations $\delta q_i, \delta t$ implies that the quantity

$$\mathcal{G} = \left( L - \sum_i \frac{\partial L}{\partial q_i} \dot{q}_i \right) \delta t + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \quad (7.4)$$

is conserved, i.e.

$$\delta S = 0 \Rightarrow \frac{d\mathcal{G}}{dt} = 0. \quad (7.5)$$

This is the so-called Noether’s Theorem.

As an example consider invariance under time translations $\delta t = \epsilon$. The conserved quantity is

$$\mathcal{G} = \left( L - \frac{\partial L}{\partial q_i} \dot{q}_i \right) \epsilon,$$
which is equivalent to the conservation of energy

\[ H = \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = p_i \dot{q}_i - L. \] (7.6)

Similarly, invariance under the rotations \( \delta x_i = \epsilon_{ijk} \varepsilon_j x_k \) gives the conserved quantity

\[ \frac{\partial L}{\partial \dot{x}_i} \epsilon_{ijk} \varepsilon_j x_k = \varepsilon \cdot (\vec{x} \times \vec{p}) \] (7.7)

which is the standard angular momentum.

### 7.2 Noether’s Theorem in Field Theory

The proof of Noether’s theorem in Field Theory goes as follows:

Consider the variations

\[ \delta \phi_a(x) = \phi'_a(x') - \phi_a(x), \quad \delta_0 \phi_a(x) = \phi'_a(x) - \phi_A(x), \quad \delta x^\mu = x'^\mu - x^\mu. \] (7.8)

Note that

\[ \delta_0 \partial_\mu \{\ldots\} = \partial_\mu \delta_0 \{\ldots\} \] (7.9)

and

\[ \delta \{\ldots\} = \delta_0 \{\ldots\} + \delta x^\mu \partial_\mu \{\ldots\}. \] (7.10)

The resulting variation in the Action is\(^1\)

\[
\delta S = \int \left\{ \delta (d^4x) \mathcal{L} + d^4x \delta \mathcal{L} \right\} \approx \int d^4x \left\{ (\partial_\mu \delta x^\mu) \mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L} + \delta_0 \mathcal{L} \right\}
\]

\[
= \int d^4x \left\{ (\partial_\mu \delta x^\mu) \mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi_a} \delta_0 \phi_a + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \delta_0 \partial_\mu \phi_a \right\}
\]

\[\approx \int d^4x \left\{ (\partial_\mu \delta x^\mu) \mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L} \right\}
\]

\[\approx \int d^4x \left\{ e^{Tr \ln(\partial_\mu x^\nu)} - 1 \right\} \approx d^4x Tr(\partial_\mu \delta x^\nu) = d^4 \partial_\mu \delta x^\mu.\]
\[
\int d^4x \left\{ \partial_\mu \left( \delta x^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a \right) + \left( \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \phi_a} \right) \right) \delta \phi_a \right\}
\]

Enforcing the equations of motion, we get

\[
\delta \mathcal{S} = \int d^4x \partial_\mu \left( \delta x^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a \right) = \int d^4x \partial_\mu J^\mu.
\]

Thus, if the Action is invariant

\[
\delta \mathcal{S} = 0 \implies \partial_\mu J^\mu = 0 \quad (7.11)
\]

if the current \( J^\mu \) is conserved. This current is called Noether Current

\[
J^\mu = \delta x^\nu \left( \mathcal{L} g^\mu_\nu - \sum_a \partial_\nu \phi_a \frac{\partial \mathcal{L}}{\partial \phi_a} \right) + \sum_a \frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a. \quad (7.12)
\]

The charges corresponding to the conserved current \( J^\mu \) are constants of the motion, i.e.

\[
Q \equiv \int d^3x J_0 \implies \frac{dQ}{dt} = \int d^3x \partial_0 J = \int d^3x \partial_\mu J^\mu = \int d^3x \vec{\nabla} \cdot \vec{J} = -\int d^3x \vec{\nabla} \cdot \vec{J} = -\oint d\vec{S} \cdot \vec{J} \rightarrow 0.
\]

7.3 The Energy-Momentum Tensor

Next, let us consider the case \( \delta \phi_a = 0 \), while \( \delta x^\mu \) corresponds to a Poincaré transformation

\[
\delta x^\mu = \epsilon^\mu \implies J^\mu = -\epsilon^\nu T^\mu_\nu, \quad (7.13)
\]

where

\[
T^\mu_\nu = -\mathcal{L} g^\mu_\nu + \sum_a \left( \partial_\nu \phi_a \right) \frac{\partial \mathcal{L}}{\partial \phi_a}. \quad (7.14)
\]
Since $\epsilon_\mu$ is a constant vector, $T^\mu_\nu$, which is called the energy-momentum tensor\(^2\), is conserved
\[ \partial_\mu T^\mu_\nu = 0 . \quad (7.15) \]

Note that
\[ T_{00} = -\mathcal{L} + \sum_a \dot{\phi}_a \frac{\partial \mathcal{L}}{\partial \phi_a} = \sum_a \dot{\phi}_a \pi_a - \mathcal{L} = \mathcal{H}, \quad (7.16) \]
which is the Hamiltonian density of the system, and
\[ T^i_0 = \sum_a \dot{\phi}_a \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}, \quad (7.17) \]
which is the field momentum density. The integrals
\[ H = \int d^3x T_{00} \quad P^i = \int d^3x T^i_0 \]
or
\[ P^\mu = \int d^3x T^\mu_0 \quad (7.18) \]
correspond to the associated (conserved) charge, i.e. the four-momentum of the system.

The energy-momentum tensor for a Hermitean scalar field $\phi(x)$ with a Lagrangean of the form
\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi g^{\mu\nu} \partial_\nu \phi - V(\phi) \]
is automatically symmetric. It comes out as
\[ T_{\mu\nu} = -\mathcal{L} g_{\mu\nu} + \partial_\mu \phi \partial_\nu \phi. \quad (7.19) \]

\(^2\)It can be shown that the above energy-momentum tensor can also be obtained by considering general spacetime metric variations $\delta g_{\mu\nu}$ of the Action
\[ S = \int d^4x \sqrt{-g} \mathcal{L}, \]
where $g = \text{Det}(g_{\mu\nu})$, as
\[ \int d^4x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} = -\delta S. \]
For a general field (vector, spinor, . . . ) the canonical energy-momentum tensor, defined as above, is not necessarily symmetric. Nevertheless, a symmetric energy-momentum tensor can always be defined by adding a suitable conserved function to it.