Chapter 8

THE COMPLEX SCALAR FIELD

A complex scalar field $\phi(x)$, that satisfies the Klein-Gordon equation,

$$\left(\Box + m^2 \right) \phi = 0 \tag{8.1}$$

is described by the Lagrange density

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi \,. \tag{8.2}$$

This Lagrangean is invariant under the continuous set of transformations

$$\phi(x) \to \phi'(x) = e^{-i\alpha}\, \phi(x), \qquad \delta\phi(x) = -i\alpha\, \phi(x)\,. \tag{8.3}$$

These transformations define a $U(1)$ group of transformations. Following the Noether procedure, the associated conserved current is

$$\mathcal{J}^\mu = i \left(\phi^\dagger \partial^\mu \phi - \phi \partial^\mu \phi^\dagger \right) . \tag{8.4}$$

The corresponding conserved charge is

$$\mathcal{Q} = \int d^3x\, \mathcal{J}_0 = i \int d^3x \left(\phi^\dagger \dot{\phi} - \dot{\phi}^\dagger \phi \right) . \tag{8.5}$$

Quantization proceeds in the standard way with the introduction of canonical momenta

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^\dagger, \quad \pi^\dagger = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger} = \dot{\phi}\,. \tag{8.6}$$

The next step is to write down the canonical commutation relations

$$
\begin{aligned}
\left[\, \phi(\vec{x},\, x_0),\, \pi(\vec{x}\,',\, x_0) \,\right] &= i\delta(\vec{x} - \vec{x}\,') \\
\left[\, \phi^\dagger(\vec{x},\, x_0),\, \pi^\dagger(\vec{x}\,',\, x_0) \,\right] &= i\delta(\vec{x} - \vec{x}\,')
\end{aligned}
\tag{8.7}
$$

while all the other equal time commutators vanish. The operator solution to the Klein-Gordon equation can be expanded in plane waves as

$$
\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left(a(k)\, e^{-ik\cdot x} + b^\dagger(k)\, e^{ik\cdot x} \right) \tag{8.8}
$$

The operators a, b satisfy standard harmonic oscillator commutation relations

$$
\begin{aligned}
&\left[\, a(k),\, a^\dagger(k') \,\right] = \left[\, b(k),\, b^\dagger(k') \,\right] = \delta(\vec{k} - \vec{k}\,') \\
&[a,\, a] = [b,\, b] = 0 \\
&[a,\, b] = \left[a,\, b^\dagger \right] = 0
\end{aligned}
\tag{8.9}
$$

Again, the *vacuum* state is defined by

$$
a(k)|0\rangle = b(k)|0\rangle \tag{8.10}
$$

as being *"being empty of particles (a) and antiparticles (b)"*. Substituting (8.8) in the expression of the Hamiltonian

$$
H = \int d^3x \left(\pi^\dagger \pi + \vec{\nabla}\phi^\dagger \cdot \vec{\nabla}\phi + \phi^\dagger \phi \right)
$$

we obtain, after considerable calculation,

$$
H = \int d^3k\, \omega(k) \left(a^\dagger(k) a(k) + b^\dagger(k) b(k) \right) . \tag{8.11}
$$

Since the charge Q is a constant of the motion its corresponding quantum operator will commute with the Hamiltonian. Thus, the energy eigenstates will also be eigenstates of Q. We may call the corresponding symmetry *"particle number"* and assign particle number +1 to the particles created by $a^\dagger$ and particle number -1 to the particles created by $b^\dagger$ (antiparticles). The expression of Q in terms of the creation/annihilation operators is (up to normal ordering)

$$Q = \int d^3k \left(a^\dagger(k)a(k) - b^\dagger(k)b(k) \right) \tag{8.12}$$

The vacuum must have vanishing particle number

$$Q|0\rangle = 0 . \tag{8.13}$$

We can proceed now proving that the following commutation relations are true

$$\begin{aligned} [\,H,\, a(k)\,] &= -\omega(k)a(k) \quad \left[H,\, a^\dagger(k)\right] = \omega(k)a^\dagger(k) \\ [\,H,\, b(k)\,] &= -\omega(k)b(k) \quad \left[H,\, b^\dagger(k)\right] = \omega(k)b^\dagger(k) \end{aligned} \tag{8.14}$$

It follows then that the state

$$|1_k,\, 1\rangle = \sqrt{2\omega}\, a^\dagger(k)|0\rangle \tag{8.15}$$

is an energy eigenstate of momentum $\vec{k}$ and energy $\omega = \sqrt{\vec{k}^2 + m^2}$ with particle number +1. Similarly, the state

$$|1_k,\, -1\rangle = \sqrt{2\omega}\, b^\dagger(k)|0\rangle \tag{8.16}$$

is an one antiparticle state of momentum k. In general,

$$|N_k,\, 1;\, N'_{k'},\, -1\rangle \propto \left(a^\dagger(k)\right)^N \left(b^\dagger(k')\right)^{N'}|0\rangle$$

is a state of N particles of momentum k and N' antiparticles of momentum k'.