Chapter 9

SPINOR FIELDS

9.1 Lorentz Transformations of Dirac Spinors

Theorem: *A set of matrices that satisfy the (Clifford-Dirac) Algebra*

$$\{\,\gamma^{\mu},\,\gamma^{\nu}\,\} \;=\; 2g^{\mu\nu}\,\mathbf{1} \tag{9.1}$$

defines the following representation of the Lorentz Algebra

$$S^{\mu\nu} \;=\; \frac{1}{2}\sigma^{\mu\nu} \;\equiv\; \frac{i}{4}\,[\,\gamma^{\mu},\,\gamma^{\nu}\,]\;. \tag{9.2}$$

In $D = 4$ Minkowski space the minimal dimension[1] of these matrices is 4.. The quantities $S_{\mu\nu}$ will prove to be the generators of a Lorentz transformation of a Dirac spinor[2].

At this point it is convenient to introduce a representation for

[1]Considering any two different matrices we have

$$Tr(\gamma^{\mu}\gamma^{\nu}) = (-1)^{D}Tr(\gamma^{\nu}\gamma^{\mu}) = (-1)^{D}Tr(\gamma^{\mu}\gamma^{\nu})$$

implying that D must be even. $D = 2$ is not possible since the space of 2×2 matrices is spanned by the unit matrix and *three* anticommuting matrices (Pauli matrices). No four 2×2 anticommuting matrices exist. Thus, the lowest dimensionality of these matrices is 4.

[2]They shouldn't be confused with the analogous generators for a vector for which we have used a similar *calligraphic* symbol $\mathcal{S}_{\rho\sigma}$.

the *Dirac Gamma Matrices*

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma_i \\ -\sigma_i & \mathbf{0} \end{pmatrix} \tag{9.3}$$

This is the *Weyl or Chiral Representation*[3]. In this representation the matrices $S^{\mu\nu}$ are

$$S^{\mu\nu} = \frac{i}{2}\gamma^\mu\gamma^\nu - \frac{i}{2}g^{\mu\nu} = \begin{cases} S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \\ S^{ij} = \frac{1}{2}\begin{pmatrix} \epsilon_{ijk}\sigma_k & 0 \\ 0 & \epsilon_{ijk}\sigma_k \end{pmatrix} \end{cases} \tag{9.4}$$

and

$$\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} = \begin{pmatrix} \frac{i}{4}(-2i\omega_{0k} + \omega_{ij}\epsilon_{ijk})\sigma_k & 0 \\ 0 & \frac{i}{4}(2i\omega_{0k} + \omega_{ij}\epsilon_{ijk})\sigma_k \end{pmatrix}$$

or

$$-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} = \begin{pmatrix} \frac{i}{2}\left(\vec{\omega} + i\vec{\nu}\right)\cdot\vec{\sigma} & 0 \\ 0 & \frac{i}{2}\left(\vec{\omega} - i\vec{\nu}\right)\cdot\vec{\sigma} \end{pmatrix} \tag{9.5}$$

where we have set

$$\nu_k = \omega_{0k}, \ \ \omega_k = -\frac{1}{2}\epsilon_{kij}\omega_{ij} \, .$$

[3]Other possible representations are *the Pauli-Dirac representation*

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma_i \\ -\sigma_i & \mathbf{0} \end{pmatrix}$$

and *the Majorana representation*

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \ \gamma^1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}, \ \gamma^2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \ \gamma^3 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}.$$

In the latter, all gamma matrices are purely imaginary. All these representasions are related to each other through a unitary transformation.

We also have that

$$\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^n = \begin{pmatrix} \left(\frac{i}{2}\left(\vec{\omega} + i\vec{\nu}\right)\cdot\vec{\sigma}\right)^n & 0 \\ 0 & \left(\frac{i}{2}\left(\vec{\omega} - i\vec{\nu}\right)\cdot\vec{\sigma}\right)^n \end{pmatrix}$$

and

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^n = \begin{pmatrix} \sum_{n=0}^{\infty}\left(\frac{i}{2}\left(\vec{\omega} + i\vec{\nu}\right)\cdot\vec{\sigma}\right)^n & 0 \\ 0 & \sum_{n=0}^{\infty}\left(\frac{i}{2}\left(\vec{\omega} - i\vec{\nu}\right)\cdot\vec{\sigma}\right)^n \end{pmatrix}$$

or

$$\Lambda_{1/2} \equiv e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} = \begin{pmatrix} e^{\frac{i}{2}(\vec{\omega}+i\vec{\nu})\cdot\vec{\sigma}} & 0 \\ 0 & e^{\frac{i}{2}(\vec{\omega}-i\vec{\nu})\cdot\vec{\sigma}} \end{pmatrix} \tag{9.6}$$

This matrix is the one appearing in the Lorentz transformation of a Dirac spinor in (2.30). Therefore, a Dirac spinor transforms as

$$\Psi'(x') = \Lambda_{1/2}\,\Psi(x)\ . \tag{9.7}$$

9.2 The Dirac Lagrangean

It is not difficult to show that the spinor

$$\overline{\Psi} \equiv \Psi^{\dagger}\,\gamma^0, \tag{9.8}$$

called the *Pauli-adjoint*, transforms under Lorentz transformations, with the inverse transformation[4]

$$\Lambda_{1/2}^{-1} = e^{\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\ .$$

Thus, the quantity $\overline{\Psi}\Psi$ is a Lorentz-scalar.

[4]Note that

$$\gamma^0\,\Lambda_{1/2}\,\gamma^0 = \Lambda_{1/2}^{-1}\,.$$

Another Lorentz scalar with one spacetime derivative can be formed as follows:

$$\overline{\Psi}\gamma^\mu \partial_\mu \Psi \,. \tag{9.9}$$

Let's consider

$$\overline{\Psi}'\gamma^\mu \partial'_\mu \Psi' \,=\, \overline{\Psi}\Lambda_{1/2}^{-1}\gamma^\mu \,\Lambda_{1/2}\frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu \Psi \,=\, \left(\Lambda^{-1}\right)_\mu^{\ \nu}\overline{\Psi}\left(\Lambda_{1/2}^{-1}\gamma^\mu \Lambda_{1/2}\right)\partial_\nu \Psi$$

It can be shown that

$$\Lambda_{1/2}^{-1}\gamma^\mu \Lambda_{1/2} \,=\, \Lambda^\mu_{\ \nu}\gamma^\nu \,. \tag{9.10}$$

To keep things simple, we show this at the infinitesimal level. We have

$$\left(1 \,+\, \frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\right)\gamma^\mu \left(1 \,-\, \frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\right) \,=\, \gamma^\mu \,+\, \frac{i}{2}\omega_{\alpha\beta}\left[\, S^{\alpha\beta},\, \gamma^\mu \,\right]$$

$$=\, \gamma^\mu \,-\, \frac{1}{2}\omega_{\alpha\beta}\left(\gamma^\alpha g^{\beta\mu} \,-\, \gamma^\beta g^{\alpha\mu}\right) \,=\, \gamma^\mu \,+\, \omega^\mu_{\ \nu}\gamma^\nu \,=\, \Lambda^\mu_{\ \nu}\gamma^\nu \,.$$

Thus, quantity (9.9) is a scalar. Since this quantity behaves in the Action as an antihermitean operator[5], an "i" is needed if such a term is part of the Action. Thus, having proven that $\overline{\Psi}\Psi$ and $i\overline{\Psi}\gamma^\mu\partial_\mu\Psi$ are Hermitean scalars, the simplest billinear Lagrange density that can be written in terms of a Dirac spinor (*Dirac Lagrangean*) is

$$\mathcal{L} \,=\, i\overline{\Psi}\gamma^\mu \partial_\mu \Psi \,-\, m\overline{\Psi}\Psi \,. \tag{9.11}$$

An equivalent expression, differing by a total divergence, is

$$\mathcal{L} \,=\, -i\partial_\mu \overline{\Psi}\,\gamma^\mu \Psi \,-\, m\overline{\Psi}\Psi \,. \tag{9.12}$$

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$$\left(\overline{\Psi}\gamma^\mu \partial_\mu \Psi\right)^\dagger \,=\, \partial_\mu \Psi^\dagger (\gamma^\mu)^\dagger \gamma^0 \Psi \,=\, \partial_\mu \overline{\Psi}\gamma^\mu \Psi$$

$$=\, -\overline{\Psi}\gamma^\mu \partial_\mu \Psi \,+\, \partial_\mu \left(\overline{\Psi}\gamma^\mu \Psi\right)$$

We have made use of the identity

$$(\gamma^\mu)^\dagger \,=\, \gamma^0 \gamma^\mu \gamma^0 \,.$$

The parameter m will turn out to be the mass of the associated particle at the quantum level. The equations of motion resulting from this Lagrangean are

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_\alpha}\right) = \frac{\partial\mathcal{L}}{\partial\psi_\alpha} \Longrightarrow i\partial_\mu\overline{\Psi}\,\gamma^\mu\Psi + m\overline{\Psi} = 0$$
$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi^\dagger_\alpha}\right) = \frac{\partial\mathcal{L}}{\partial\Psi_\alpha} \Longrightarrow i\gamma^\mu\partial_\mu\Psi - m\Psi = 0 \tag{9.13}$$

The *Dirac Lagrangean* of a Dirac spinor can be written in terms of the two left-handed Weyl spinors ψ and χ that compose the Dirac spinor $\Psi(x)$. First, let's introduce the matrices

$$\sigma^\mu = (\mathbf{1},\,\sigma_i),\quad \overline{\sigma}^\mu = (\mathbf{1},\,-\sigma_i) \Longrightarrow \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \tag{9.14}$$

Substituting

$$\Psi = \begin{bmatrix} \psi \\ i\sigma_2\chi^* \end{bmatrix} \Longrightarrow \mathcal{L} = i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi + i\chi^\dagger\overline{\sigma}^\mu\partial_\mu\chi - m\psi\cdot\chi + h.c.$$

where $\psi\cdot\chi = \psi_a\epsilon_{ab}\chi_b = i\psi^\perp\sigma_2\chi$. We have used the fact that the spinors are anticommuting. Note also the identity

$$\sigma_2\sigma^\mu\sigma_2 = (\overline{\sigma}^\mu)^\perp\,.$$

The resulting equations are the so-called *Weyl equations* and are only coupled through the mass term

$$i\overline{\sigma}^\mu\partial_\mu\psi_L = mi\sigma_2\chi_L^* = m\chi_R$$
$$i\overline{\sigma}^\mu\partial_\mu\chi_L = mi\sigma_2\psi_L^* = m\psi_R \tag{9.15}$$

The left and right-handed Weyl spinors correspond to left and right-handed Dirac spinors, which are defined in terms of the γ_5 matrix

$$\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \Longrightarrow \{\,\gamma_5,\,\gamma^\mu\,\} = 0\,. \tag{9.16}$$

In terms of γ_5 we have

$$\Psi_L \equiv \frac{1}{2}(1 + \gamma_5)\Psi = \begin{bmatrix} \psi_L \\ 0 \end{bmatrix} \tag{9.17}$$

$$\Psi_R \equiv \frac{1}{2}(1 - \gamma_5)\Psi = \begin{bmatrix} 0 \\ \chi_R \end{bmatrix}. \tag{9.18}$$

In terms of Ψ_L and Ψ_R the Dirac Lagrangean is written as

$$\mathcal{L} = i\overline{\Psi}_L\gamma^\mu\partial_\mu\Psi_L + i\overline{\Psi}_R\gamma^\mu\partial_\mu\Psi_R - m\overline{\Psi}_L\Psi_R - m\overline{\Psi}_R\Psi_L\,. \tag{9.19}$$

It is possible to have a Dirac spinor that is defined in terms of a single Weyl spinor. Such a spinor has half of the degrees of freedom of a genuine Dirac spinor and it is called a *Majorana spinor*

$$\Psi = \begin{bmatrix} \psi \\ i\sigma_2\psi^* \end{bmatrix}. \tag{9.20}$$

The Dirac Lagrangean possesses the following continuous global symmetry

$$\Psi(x) \to \Psi'(x) = e^{i\alpha}\,\Psi(x) \Longrightarrow \delta\Psi = i\alpha\,\Psi \tag{9.21}$$

Following the Noether procedure, we obtain the conserved current

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi}\,\delta\Psi + \delta\Psi^\dagger\frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi^\dagger} = -2\alpha\,\overline{\Psi}\gamma^\mu\Psi \Longrightarrow \mathcal{J}^\mu = \overline{\Psi}\gamma^\mu\Psi\,. \tag{9.22}$$

The corresponding charge is

$$Q = \int d^3x\,\Psi^\dagger\Psi\,.$$

This conserved current can be interpreted as a *“Fermion Number”* current, associating *charge* +1 to the spinor Ψ and -1 to the conjugate spinor $\Psi^\dagger$.

Before closing this section, it is useful to consider the Hamiltonian density corresponding to the above Dirac Lagrangean. The canonical momentum corresponding to Ψ is

$$\varpi \,=\, \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} \,=\, i\Psi^{\dagger}\,. \tag{9.23}$$

The Hamiltonian density is

$$\mathcal{H} \,=\, \varpi\dot{\Psi} \,-\, \mathcal{L} \,=\, i\Psi^{\dagger}\dot{\Psi} \,-\, i\Psi^{\dagger}\dot{\Psi} \,-\, i\overline{\Psi}\vec{\gamma}\cdot\vec{\nabla}\Psi \,+\, m\overline{\Psi}\Psi$$

or

$$\mathcal{H} \,=\, -i\overline{\Psi}\vec{\gamma}\cdot\vec{\nabla}\Psi \,+\, m\overline{\Psi}\Psi\,. \tag{9.24}$$

The Hamilton density is part of the energy-momentum tensor which can be constructed in a standard fashion following the Noether formula

$$T^{\mu}_{\ \nu} \,=\, -\mathcal{L}\,g^{\mu}_{\ \nu} \,+\, \partial_{\nu}\Psi\,\frac{\partial \mathcal{L}}{\partial\partial_{\mu}\Psi}$$

and obtain

$$T^{\mu}_{\ \nu} \,=\, -\mathcal{L}\,g^{\mu}_{\ \nu} \,+\, i\partial_{\nu}\Psi\,(-)\Psi^{\dagger}\gamma^{0}\gamma^{\mu}$$

or

$$T^{\mu}_{\ \nu} \,=\, -\left(i\overline{\Psi}\gamma^{\rho}\partial_{\rho}\Psi \,-\, m\overline{\Psi}\Psi\right)\,g^{\mu}_{\ \nu} \,+\, i\overline{\Psi}\gamma^{\mu}\partial_{\nu}\Psi\,. \tag{9.25}$$

The four-momentum density is

$$\mathcal{P}^{\mu} \,=\, T^{\mu}_{\ 0} \,=\, \begin{cases} \mu=0 \implies \mathcal{H} \,=\, -i\overline{\Psi}\vec{\gamma}\cdot\vec{\nabla}\Psi \,+\, m\overline{\Psi}\Psi \\[2ex] \mu=j \implies \mathcal{P}^{j} \,=\, i\overline{\Psi}\gamma^{j}\dot{\Psi} \end{cases} \tag{9.26}$$

9.3 Plane Wave Spinors

By *plane wave solutions* of the Dirac equation we mean solutions of the form

$$\begin{aligned} \Psi^{(+)}(x) \,&=\, u(k)\,e^{-ik\cdot x} \\ \Psi^{(-)}(x) \,&=\, v(k)\,e^{ik\cdot x} \end{aligned}\,. \tag{9.27}$$

The $u(k)$'s are referred to as *positive-frequency waves*, while the $v(k)$'s as *negative-frequency waves*, since the former evolve in time with the standard quantum mechanical exponential $e^{-i\omega t}$, while the latter evolve with $e^{+i\omega t}$. Substituting into the Dirac equation, we obtain

$$\begin{aligned} (\,\gamma^\mu k_\mu - m)\, u(k) &= 0 \\ (\,\gamma^\mu k_\mu + m\,)\, v(k) &= 0 \end{aligned} \tag{9.28}$$

Acting on them with $(\,\gamma^\mu k_\mu \pm m)$, we obtain further

$$\left(k^2 - m^2\right) u(k) = \left(k^2 - m^2\right) v(k) = 0\,.$$

Therefore, k^μ must obey the relativistic energy-momentum relation, or to be *"on (mass-) shell"*

$$k^2 = m^2 \Longleftrightarrow k_0^2 - (\vec{k})^2 = m^2\,. \tag{9.29}$$

In what follows we shall always take $k_0 > 0$. The positive frequency solutions $\Psi^{(+)}$, defined by

$$\begin{pmatrix} -m & k_0 - \vec{\sigma}\cdot\vec{k} \\ k_0 + \vec{\sigma}\cdot\vec{k} & -m \end{pmatrix} u(k) = 0 \tag{9.30}$$

having the correct quantum mechanical dependence on time, are obviously interpreted as particles of mass m, energy $k_0 > 0$ and momentum $\vec{k}$. In contrast, the negative frequency solutions $\Psi^{(-)}$

$$\begin{pmatrix} m & k_0 - \vec{\sigma}\cdot\vec{k} \\ k_0 + \vec{\sigma}\cdot\vec{k} & m \end{pmatrix} v(k) = 0 \tag{9.31}$$

having the *"wrong"* dependence on time, are in need of interpretation.

In the *rest frame* of the particle, where $\vec{k} = 0$ and $k_0 = m$, these equations look particularly simple and they can be solved trivially. Namely, they give

$$
\begin{pmatrix} -1 & 1 \\ & \\ 1 & -1 \end{pmatrix} u(k_0) \,=\, 0 \Longrightarrow u(k_0) \,=\, N_0 \begin{pmatrix} \xi \\ \\ \xi \end{pmatrix}
$$

$$
\begin{pmatrix} 1 & 1 \\ & \\ 1 & 1 \end{pmatrix} v(k_0) \,=\, 0 \Longrightarrow v(k_0) \,=\, \tilde{N}_0 \begin{pmatrix} \tilde{\xi} \\ \\ -\tilde{\xi} \end{pmatrix} \tag{9.32}
$$

where ξ and $\tilde{\xi}$ are arbitrary two-component spinors and N_0, $\tilde{N}_0$ normalization constants. The spinors ξ, $\tilde{\xi}$ can be chosen to be any normalized

$$
\xi^\dagger \xi \,=\, \tilde{\xi}^\dagger \tilde{\xi} \,=\, 1
$$

combination of the linearly independent spinors

$$
\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
$$

Considering pairs of orthonormal two-component spinors

$$
\xi_a^\dagger \xi_b \,=\, \tilde{\xi}_a^\dagger \tilde{\xi}_b \,=\, \delta_{ab} \tag{9.33}
$$

we have corresponding orthogonal spinors $u^{(a)}(k)$, $v^{(a)}(k)$.

The general spinor $u(k)$ can be expressed as[6]

$$
u(k) \,=\, N\, (\,\gamma^\mu k_\mu \,+\, m)\; u(k_0) \tag{9.34}
$$

since we automaticall have $(\,\gamma^\mu k_\mu \,-\, m)\, u(k) \,=\, N\, (\,k^2 \,-\, m^2)\, u(k_0) \,=\, 0$. This amounts to

$$
u(k) \,=\, N \begin{pmatrix} \left(E + m - \vec{\sigma}\cdot\vec{k}\right)\xi \\ \\ \left(E + m + \vec{\sigma}\cdot\vec{k}\right)\xi \end{pmatrix}. \tag{9.35}
$$

Assuming the covariant normalization

$$
\overline{u}^{(a)}(k)\, u^{(b)}(k) \,=\, 2m\delta^{ab}\,, \tag{9.36}
$$

[6]We have absorbed the normalization constant N_0 in the new overall normalization constant N.

we obtain

$$N = \frac{1}{\sqrt{2(E+m)}} .$$

Thus, we have

$$u^{(a)}(k) = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} \left(E + m - \vec{\sigma}\cdot\vec{k}\right)\xi_a \\ \\ \left(E + m + \vec{\sigma}\cdot\vec{k}\right)\xi_a \end{pmatrix} . \tag{9.37}$$

In an analogous way, for the *"negative energy"* solutions, assuming the normalization condition,

$$\overline{v}^{(a)}(k)v^{(b)}(k) = -2m\delta_{ab} , \tag{9.38}$$

we have

$$v^{(a)}(k) = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} \left(-m - E + \vec{\sigma}\cdot\vec{k}\right)\tilde{\xi}_a \\ \\ \left(m + E + \vec{\sigma}\cdot\vec{k}\right)\tilde{\xi}_a \end{pmatrix} . \tag{9.39}$$

Note that the orthogonality condition

$$\overline{u}^{(a)}(k)\, v^{(b)}(k) = 0$$

is satisfied automatically.

9.3.1 An alternative (tedious but instructive) way to obtain the general plane wave spinor

A boost from an inertial frame Σ, where the plane wave spinor corresponds to a particle of four-momentum $p^\mu = (E, 0, 0, p)$ to the rest frame Σ', where the four-momentum is $p'^\mu = (m, 0, 0, 0)$, is

$$\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

We have parametrised the boost in terms of the *rapidity* $\eta \equiv \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)$. We have

$$\tanh\eta \,=\, \beta\,, \qquad \gamma \,=\, \cosh\eta \tag{9.40}$$

and

$$E \,=\, m\,\cosh\eta, \qquad p \,=\, m\,\sinh\eta\,. \tag{9.41}$$

An infinitesimal Lorentz transformation

$$\Lambda^{\mu}_{\ \nu} \,\approx\, \delta^{\mu}_{\nu} \,+\, \omega^{\mu}_{\ \nu} \,\Longrightarrow\, \left(\begin{array}{cc} 1 & -\eta \\ -\eta & 1 \end{array}\right) \,+\, O(\eta^2)$$

gives

$$\omega^{0}_{\ 3} \,=\, \omega^{3}_{\ 0} \,=\, -\eta\,.$$

The corresponding transformation on the spinor reads

$$\mathcal{S} \,=\, e^{\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}} \,=\, e^{\frac{i}{2}\eta\,S_{03}} \,=\, e^{-\frac{\eta}{2}\left(\begin{array}{cc} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{array}\right)} \,=$$

$$\left(\begin{array}{cc} e^{-\frac{\eta}{2}\sigma_3} & 0 \\ 0 & e^{\frac{\eta}{2}\sigma_3} \end{array}\right) \,=\, \left(\begin{array}{cc} \cosh(\eta/2)\,-\sinh(\eta/2)\,\sigma_3 & 0 \\ 0 & \cosh(\eta/2)\,+\,\sinh(\eta/2)\,\sigma_3 \end{array}\right)\,.$$

Note that

$$\begin{array}{l} \cosh\eta \,=\, 2\cosh^2(\eta/2)\,+\,1 \\ \sinh\eta \,=\, 2\sinh(\eta/2)\cosh(\eta/2) \end{array} \;\Longrightarrow\; \begin{array}{l} \cosh(\eta/2) \,=\, \sqrt{\frac{E-m}{2m}} \\ \sinh(\eta/2) \,=\, \frac{p}{m}\sqrt{\frac{2m}{E-m}} \end{array} \tag{9.42}$$

Thus, we end up with

$$u(p) \,=\, N\left(\begin{array}{cc} \cosh(\eta/2)-\sinh(\eta/2)\sigma_3 & 0 \\ 0 & \cosh(\eta/2)\,+\,\sinh(\eta/2)\sigma_3 \end{array}\right)\left(\begin{array}{c} \xi \\ \xi \end{array}\right)$$

Similarly we may obtain

$$v(p) \,=\, \tilde{N}\left(\begin{array}{cc} \cosh(\eta/2)\,+\,\sinh(\eta/2)\sigma_3 & 0 \\ 0 & \cosh(\eta/2)\,-\sinh(\eta/2)\sigma_3 \end{array}\right)\left(\begin{array}{c} \tilde{\xi} \\ -\tilde{\xi} \end{array}\right)$$

or

$$u(p) \,=\, N \begin{pmatrix} (\cosh(\eta/2) - \sinh(\eta/2)\sigma_3)\,\xi \\ \\ (\cosh(\eta/2) \,+\, \sinh(\eta/2)\sigma_3)\,\xi \end{pmatrix} \qquad (9.43)$$

and

$$v(p) \,=\, \tilde{N} \begin{pmatrix} (\cosh(\eta/2) \,+\, \sinh(\eta/2)\sigma_3)\,\tilde{\xi} \\ \\ -\,(\cosh(\eta/2) \,-\, \sinh(\eta/2)\sigma_3)\,\tilde{\xi} \end{pmatrix} \qquad (9.44)$$

At this point we note the identity

$$(\cosh(\eta/2) \,-\, \sigma_3 \sinh(\eta/2))^2 \,=\, \cosh\eta - \sigma_3 \sinh\eta \,=\, m^{-1}\,(E \,-\, \sigma_3 p) \,=\, m^{-1}\,\sigma^\mu p_\mu \qquad (9.45)$$

and

$$(\cosh(\eta/2) \,+\, \sigma_3 \sinh(\eta/2))^2 \,=\, \cosh\eta + \sigma_3 \sinh\eta \,=\, m^{-1}\,(E \,+ \sigma_3 p) \,=\, m^{-1}\,\overline{\sigma}^\mu p_\mu \qquad (9.46)$$

where

$$\sigma^\mu \,=\, (\mathbf{1},\, \vec{\sigma})\,, \;\; \overline{\sigma}^\mu \,=\, (\mathbf{1},\, -\vec{\sigma})\;.$$

Thus, we may write the plane wave spinor in a compact form[7]

$$u(p) \,=\, \begin{pmatrix} (p\cdot\sigma)^{1/2}\xi \\ \\ (p\cdot\overline{\sigma})^{1/2}\xi \end{pmatrix}\,. \qquad (9.47)$$

We have imposed the normalization condition

$$\overline{u}(p)u(p) \,=\, 2m$$

[7]Equivalence to the previously obtained form, namely

$$(\sigma\cdot k)^{1/2} \,=\, \frac{E+m-\vec{\sigma}\cdot\vec{k}}{\sqrt{2(E+m)}}$$

can be shown in a straightforward fashion by taking the square. In an analogous fashion we have

$$(\overline{\sigma}\cdot k)^{1/2} \,=\, \frac{E+m+\vec{\sigma}\cdot\vec{k}}{\sqrt{2(E+m)}}$$

and made use of the identity

$$(\sigma \cdot p)(\overline{\sigma} \cdot p) \,=\, p^2 \,=\, m^2\,. \tag{9.48}$$

In an analogous fashion we obtain

$$v(p) \,=\, \begin{pmatrix} (p\cdot\overline{\sigma})^{1/2}\tilde{\xi} \\ \\ -(p\cdot\sigma)^{1/2}\tilde{\xi} \end{pmatrix} \tag{9.49}$$

We have imposed the normalization condition

$$\overline{v}(p)v(p) \,=\, -2m\,.$$

Before closing this section we shall consider a set of useful identities satisfied by the spinors $u(k)$ and $v(k)$. Their proof is straightforward. They are

$$\begin{aligned}
&(u^{(a)}(k))^{\dagger}u^{(b)}(k) \,=\, 2E\delta_{ab}\\
&(v^{(a)})^{\dagger}v^{(b)}(k) \,=\, 2E\delta_{ab}\\
&(u^{(a)}(k))^{\dagger}v^{(b)}(k) \,\neq\, 0\\
&(v^{(a)}(k))^{\dagger}u^{(b)}(k) \,\neq\, 0\\
&(u^{(a)}(k))^{\dagger}v^{(b)}(\tilde{k}) \,=\, 0\\
&(v^{(a)}(k))^{\dagger}u^{(b)}(\tilde{k}) \,=\, 0
\end{aligned} \tag{9.50}$$

where

$$\tilde{k}^{\mu} \,=\, (E,\, -\vec{k})\,.$$

In addition, we have the *completeness relations*

$$\begin{aligned}
&\textstyle\sum_s u_{\alpha}^{(s)}(k)\overline{u}_{\beta}^{(s)}(k) \,=\, (\gamma\cdot k\,+\,m)_{\alpha\beta}\\
&\textstyle\sum_s v_{\alpha}^{(s)}(k)\overline{v}_{\beta}^{(s)}(k) \,=\, (\gamma\cdot k\,-\,m)_{\alpha\beta}
\end{aligned} \tag{9.51}$$

9.4 A Note on Helicity

The operator

$$h \,=\, \hat{\mathbf{p}}\cdot\vec{S}\,, \tag{9.52}$$

with

$$S_i \,=\, \frac{1}{2}\epsilon_{ijk}S_{jk} = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix},$$

is called *helicity*.

Let's consider the spinor $u(p)$ with the momentum along the $\hat{z}$ axis

$$u^{(a)}(p) \,=\, \frac{1}{\sqrt{2(E+m)}}\begin{pmatrix} (E+m-\sigma_3 p)\xi^{(a)} \\ (E+m+\sigma_3 p)\xi^{(a)} \end{pmatrix}. \tag{9.53}$$

We make the particular choice of the two-component spinors

$$\xi^{(1)} \,=\, \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In the limit of very large momentum (*massless limit*), we have

$$u^{(1)}(p) \,\approx\, \begin{pmatrix} 0 \\ \sqrt{2E}\,\xi^{(1)} \end{pmatrix},\quad u^{(2)}(p) \,=\, \begin{pmatrix} \sqrt{2E}\,\xi^{(2)} \\ 0 \end{pmatrix} \tag{9.54}$$

and the spinors $u^{(1)}(p)$, $u^{(2)}(p)$ are eigenstates of helicity $+1/2$, $-1/2$

$$\begin{aligned} h\,u^{(1)}(p) \,&=\, \tfrac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} u^{(1)}(p) \,\approx\, u^{(1)}(p) \\ h\,u^{(2)}(p) \,&=\, \tfrac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} u^{(2)}(p) \,\approx\, -u^{(2)}(p) \end{aligned}\,. \tag{9.55}$$

Analogously for the spinor $v(p)$. Note that in the massless case helicity is a good quantum number since no Lorentz transformation from the rest frame can be defined. A spinor of helicity $+1/2$ is called a *right-handed* spinor, while a spinor of helicity $-1/2$ is called a *left-handed* one.

In the case of massless Weyl spinors (or Weyl spinors in the limit of vanishing mass or high momentum), Weyl's equation takes the form

$$\overline{\sigma}^{\mu}\partial_{\mu}\psi_L \,=\, 0 \Longrightarrow p\tilde{\psi}_L(p) \,+\, (\vec{\sigma}\cdot\vec{p})\,\tilde{\psi}_L(p) \,=\, 0$$

or

$$h\,\tilde{\psi}_L \,=\, \frac{1}{2}\tilde{\psi}_L\,. \tag{9.56}$$

In an analogous fashion we have $h\tilde{\chi}_R \,=\, -\frac{1}{2}\tilde{\chi}_R$. This justifies the names *left* and *right-handed* spinor.