Chapter 10

QUANTIZED DIRAC FIELDS

The first step in the quantization of the Dirac field is to impose equal time commutation relations among the operator field variables. In fact this is not exactly possible, due to the Grassman nature of spinor fields, and the correct procedure corresponds to imposing *anti-commutation relations*. Since the canonical momentum is

$$\varpi(x) \,=\, i\psi^\dagger(x)\,, \tag{10.1}$$

we postulate

$$\begin{aligned} \Big\{\, \psi_\alpha(\vec{x},\, x_0),\, \psi^\dagger_\beta(\vec{x}\,',\, x_0) \,\Big\} \,&=\, \delta_{\alpha\beta}\delta(\vec{x}\,-\vec{x}\,') \\ \{\, \psi_\alpha(\vec{x},\, x_0),\, \psi_\beta(\vec{x}\,',\, x_0) \,\} \,&=\, 0 \end{aligned} \tag{10.2}$$

Expanding the general operator solution of the Dirac equation in plane wave spinors, we write

$$\psi(x) \,=\, \sum_s \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E}} \Big(a^{(s)}(k)u^{(s)}(k)e^{-ik\cdot x} \,+\, (b^{(s)}(k))^\dagger v^{(s)}(k)e^{ik\cdot x}\Big) \tag{10.3}$$

where $E \,=\, \sqrt{m^2+\vec{k}^2}$. Enforcing the anticommutation relations on ψ, implies the following anticommutation relations on the operators

a, b

$$\left\{ a^{(s)}(k),\, (a^{(s')}(k'))^\dagger \right\} = \left\{ b^{(s)}(k),\, (b^{(s')}(k'))^\dagger \right\} = \delta_{ss'}\delta(\vec{k} - \vec{k}\,')$$

$$\left\{ a^{(s)}(k),\, a^{(s')}(k') \right\} = \left\{ b^{(s)}(k),\, b^{(s')}(k') \right\} = 0$$

$$\left\{ a^{(s)}(k),\, b^{(s')}(k') \right\} = \left\{ a^{(s)}(k),\, (b^{(s')}(k'))^\dagger \right\} = 0\,. \tag{10.4}$$

Substituting the expansion (10.3) into the Hamiltonian

$$H = \int d^3x \left(-i\overline{\psi}\vec{\gamma} \cdot \vec{\nabla}\psi + m\overline{\psi}\psi \right)$$

we obtain

$$H = \sum_s \int d^3k\, E \left((a^{(s)}(k))^\dagger a^{(s)}(k) - b^{(s)}(k)(b^{(s)}(k))^\dagger \right) . \tag{10.5}$$

Anticommuting the b-operators, we obtain a normal-ordered expression plus the infinite constant $2\delta(0)\int d^3kE$, which we drop and have

$$H = \sum_s \int d^3k\, E \left((a^{(s)}(k))^\dagger a^{(s)}(k) + (b^{(s)}(k))^\dagger b^{(s)}(k) \right) . \tag{10.6}$$

From this expression we see that a state $|0\rangle$, defined by

$$a^{(s)}(k)|0\rangle = b^{(s)}(k)|0\rangle = 0 \tag{10.7}$$

is the lowest energy eigenstate, i.e. the vacuum state, having zero energy.

The commutation relations

$$\left[H,\, a^{(s)}(k) \right] = -Ea^{(s)}(k)$$

$$\left[H,\, (a^{(s)}(k))^\dagger \right] = E(a^{(s)}(k))^\dagger$$

$$\left[H,\, b^{(s)}(k) \right] = -Eb^{(s)}(k) \tag{10.8}$$

$$\left[H,\, (b^{(s)}(k))^\dagger \right] = E(b^{(s)}(k))^\dagger$$

imply that the states

$$(a^{(s)}(k))^{\dagger}|0\rangle, \quad (b^{(s)}(k))^{\dagger}|0\rangle \tag{10.9}$$

are *one-particle energy-eigenstates* of energy $E = \sqrt{\vec{k}^2 + m^2}$ and *polarization (spin) s*. The operators $a^{(s)}(k)$, $(a^{(s)})^{\dagger}$ are annihilation and creation operators of particles of the *"a-type"*, from now on, named as *particles*, while the operators $b^{(s)}(k)$, $(b^{(s)})^{\dagger}$ are annihilation and creation operators of particles of the *"b-type"*, from now on, named as *antiparticles*. Note however that two-particle or two-antiparticle states of the same momentum and spin do not exist since

$$(a^{(s)}(k))^{\dagger}(a^{(s)}(k))^{\dagger}|0\rangle \,=\, (b^{(s)}(k))^{\dagger}(b^{(s)}(k))^{\dagger}|0\rangle \,=\, 0 \tag{10.10}$$

due to the anticommutation relations, which imply that

$$\left((a^{(s)}(k))^{\dagger} \right)^2 \,=\, \left((b^{(s)}(k))^{\dagger} \right)^2 \,=\, 0\,. \tag{10.11}$$

This is an equivalent statement of the *Pauli Exclusion Principle*. The automatic antisymmetry of any multiparticle state

$$|k_1, s_1;\, \dots\, k_i, s_i,\, \dots, k_j, s_j;\, \dots\rangle \,=\, -|k_1, s_1;\, \dots\, k_j, s_j,\, \dots, k_i, s_i;\, \dots\rangle$$

is a direct consequence of the fact that we have imposed anticommutation relations instead of commutation ones. At a mathematical level, this is attributed to the Grassmann number nature of spinor fields.

The normalization of one-particle states can be chosen in an analogous fashion as in the case of the scalar field

$$|k, s\rangle \,=\, \sqrt{2E}(a^{(s)}(k))^{\dagger}|0\rangle \implies \langle k, s|k', s'\rangle \,=\, 2E\,\delta_{ss'}\,\delta(\vec{k} - \vec{k}\,') \tag{10.12}$$

Next, let's consider the unitary operator that represents Lorentz transformation on the states, namely, $U(\Lambda)$. The expression for the Lorentz transformed spinor field reads

$$U\psi(x)U^{\dagger} \,=\, \sum_s \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E}} \left(Ua^{(s)}(k)U^{\dagger}u^{(s)}(k)e^{-ik\cdot x} \right.$$

$$+U(b^{(s)}(k))^{\dagger}U^{\dagger}v^{(s)}(k)e^{ik\cdot x}\Big) \ .$$

The anticommutation relations satisfied by a, b imply that, under the transformation $k^{\mu\prime} = \Lambda^{\mu}_{\ \nu}k^{\nu}$,

$$\Big\{\, a^{\dagger}(k_1),\, a(k_2)\,\Big\} \;=\; \delta(\vec{k}_1 - \vec{k}_2) \Longrightarrow \Big\{\, a'^{\dagger}(k_1'),\, a'(k_2')\,\Big\} \;=\; \delta(\vec{k}_1{}' - \vec{k}_2{}')$$

$$\Big\{\, U\mathbf{a}^{\dagger}(k_1)U^{\dagger},\, Ua(k_2)U^{\dagger}\,\Big\} \;=\; \delta(\vec{k} - \vec{k}\,') \;=\; \frac{E_1'}{E}\delta(\vec{k}_1' - \vec{k}_2')$$

leading to

$$\begin{aligned} \sqrt{E}Ua(k)\mathbf{U}^{\dagger} \;&=\; \sqrt{E'}a'(k') \\ \sqrt{E}U(a(k))^{\dagger}U^{\dagger} \;&=\; \sqrt{E'}(a'(k'))^{\dagger} \end{aligned} \tag{10.13}$$

Returning to the Lorentz-transformed spinor field $\psi(x)$, we have

$$U\psi(x)U^{\dagger} \;=\; \sum_s \int \frac{d^3k}{(2\pi)^{3/2}2E}\Big(\sqrt{2E'}a'^{(s)}(k')u^{(s)}(k)e^{-ik\cdot x} \;+\; \sqrt{2E'}(b^{(s)}(k'))^{\dagger}v^{(s)}(k)e^{ik\cdot x}\Big)$$

$$= \sum_s \int \frac{d^3k'}{(2\pi)^{3/2}2E'}\Big(\sqrt{2E'}a'^{(s)}(k')u^{(s)}(k)e^{-ik'\cdot x'} \;+\; \sqrt{2E'}(b^{(s)}(k'))^{\dagger}v^{(s)}(k)e^{ik'\cdot x'}\Big)$$

$$= \sum_s \int \frac{d^3k'}{(2\pi)^{3/2}\sqrt{2E'}}\Big(a'^{(s)}(k')u^{(s)}(k)e^{-ik'\cdot x'} \;+\; (b^{(s)}(k'))^{\dagger}v^{(s)}(k)e^{ik'\cdot x'}\Big)\ ,$$

where $x'^{\mu} \;=\; \Lambda^{\mu}_{\ \nu}x^{\nu}$. At this point, we note that

$$u^{(s)}(k) \;=\; \Lambda^{-1}_{1/2}u^{(s)}(k')\,, \tag{10.14}$$

where the Lorentz transformation matrix of a Dirac spinor is given in (9.6). Thus, finally, we have

$$U\psi(x)U^{\dagger} \;=\; \Lambda^{-1}_{1/2}\psi(x')\,. \tag{10.15}$$

As an application of the above let's consider an infinitesimal spatial rotation. Taking

$$\Lambda^{i}_{\ j} \;=\; \delta^{i}_{j} \,+\, \omega^{i}_{\ j}, \quad U \;=\; 1 \,+\, \frac{i}{2}\omega^{ij}J_{ij}, \quad \Lambda_{1/2} \;=\; 1 \,-\, \frac{i}{2}\omega^{ij}S_{ij}$$

Substituting in (10.15) we obtain

$$[\, J_{ij},\, \psi(x)\,] \;=\; (\, S_{ij} \,+\, \mathcal{L}_{ij}\,)\,\psi(x) \tag{10.16}$$

with $\mathcal{L}_{ij} = i(x_i\nabla_j - x_i\nabla_i)$. Introducing the *spin* and *orbital angular momentum*

$$S_{ij} \,=\, \epsilon_{ijk}\begin{pmatrix} \frac{1}{2}\sigma_k & 0 \\ 0 & \frac{1}{2}\sigma_k \end{pmatrix} \equiv \epsilon_{ijk}S_k, \qquad S_k \,=\, \frac{1}{2}\epsilon_{kij}S^{ij}$$

and

$$\mathcal{L}_{ij} \,=\, -i\epsilon_{ijk}\left(\vec{x}\times\vec{\nabla}\right) \equiv \epsilon_{ijk}L_k, \quad L_k \,=\, \frac{1}{2}\epsilon_{kij}\mathcal{L}^{ij}$$

we can write

$$[\, J_{ij},\, \psi(x)\,] \,=\, \epsilon_{ijk}\,(\, S_k \,+\, L_k\,)\,\psi(x) \,=\, \epsilon_{ijk}\left(\begin{pmatrix} \frac{1}{2}\sigma_k & 0 \\ 0 & \frac{1}{2}\sigma_k \end{pmatrix} + \left(\vec{x}\times(-i\vec{\nabla})\right)\right)\psi(x)\,. \tag{10.17}$$

The operator $J_{\mu\nu}$ can be obtained if we follow the Noether procedure. It turns out that its spatial components J_{ij} are

$$J_{ij} \,=\, \epsilon_{ijk}\int d^3x\,\psi^\dagger(x)\left(\,S_k \,+\, \left(\vec{x}\times(-i\vec{\nabla})\right)\right)\psi(x) \tag{10.18}$$

Substituting (10.18) into (10.17) the latter is immediately verified.

It can be easily seen that the Dirac Lagrangean is invariant under the continuous set of transformations

$$\psi(x) \,\to\, \psi'(x) \,=\, e^{i\beta}\,\psi(x)\,. \tag{10.19}$$

The corresponding conserved *Noether current* is

$$\mathcal{J}^\mu \,=\, \overline{\psi}(x)\gamma^\mu\psi(x) \Longrightarrow \partial_\mu\mathcal{J}^\mu \,=\, 0\,. \tag{10.20}$$

The associated conserved charge is

$$Q \,=\, \int d^3x\,\psi^\dagger(x)\psi(x) \,=\, \sum_s\int d^3k\,\left(\,(a^{(s)}(k))^\dagger a^{(s)}(k) \,-\, (b^{(s)}(k))^\dagger b^{(s)}(k)\,\right)\,. \tag{10.21}$$

Acting with this charge on the one-particle states, we obtain

$$Q(a^{(s)}(k))^\dagger|0\rangle \,=\, (a^{(s)}(k))^\dagger|0\rangle$$

$$Q(b^{(s)}(k))^\dagger|0\rangle \,=\, -(b^{(s)}(k))^\dagger|0\rangle$$

Introducing the name *"Fermion Number"* for the conserved charge, we see that the one-particle states generated by $a^\dagger$ have fermion number $+1$, while the one-antiparticle states generated by $b^\dagger$ have fermion number -1. Furthermore, we can label these states as follows

$$\begin{aligned} |k,\, s,\, +1\rangle &\equiv \sqrt{2E}(a^{(s)}(k))^\dagger|0\rangle \\ |k,\, s,\, -1\rangle &\equiv \sqrt{2E}(b^{(s)}(k))^\dagger|0\rangle \end{aligned} \tag{10.22}$$

Closing this section let's consider the quantity that is analogous to the Feynman propagator of the scalar field defined by (6.9), namely

$$S_F(x-y) \;\equiv\; \langle 0|T\left(\psi(x)\,\overline{\psi}(y)\right)|0\rangle\,. \tag{10.23}$$

It is straightforward to check that

$$(\,i\gamma\cdot\partial - m\,)\,S_F(x-y) \;=\; i\delta(x-y)\,. \tag{10.24}$$

It is interesting to note that this equation is satisfied by

$$S_F(x-y) \;=\; i\,(\,i\gamma\cdot\partial \,+\, m\,)\,D_F(x-y)\,. \tag{10.25}$$

Using the momentum space expression of $D_F(x-y)$, this becomes

$$S_F(x-y) \;=\; i\int \frac{d^4k}{(2\pi)^4}\frac{\gamma^\mu k_\mu + m}{k^2-m^2+i\epsilon}\,. \tag{10.26}$$