Chapter 11

DISCRETE LORENTZ TRANSFORMATIONS

In addition to the continuous Lorentz transformations $x'^{\mu} = \Lambda^{\mu}_{\ \nu}\, x^{\nu}$, there are two additional distinct spacetime transformations that are not characterized by any continuous parameter, being, therefore, discrete. These are:

a) the *Spatial Reflection or Parity* $\mathcal{P} : (x^0,\, x^i) \Longrightarrow (x^0,\, -x^i)$

and

b) the *Time Reversal* $\mathcal{T} : (x^0,\, x^i) \Longrightarrow (-x^0,\, x^i)$.
Including these transformations into the Lorentz group defines the full Lorentz group composed of four disconnected pieces. Note however that, although, any relativistic theory has to be invariant under the standard Lorentz transformations, it need not be so under the extended group. The associated terminology refers to the standard continuous Lorentz transformations $\Lambda^{\mu}_{\ \nu}$ as *proper and orthochronous* $\mathbf{L}_{\uparrow}^{(+)}$, while the transformations $\mathcal{P}\Lambda$ are *improper, orthochronous* $\mathbf{L}_{\uparrow}^{(-)}$, $\mathcal{T}\Lambda$ are *proper, non-orthochronous*$\mathbf{L}_{\downarrow}^{(+)}$ and $\mathcal{P}\mathcal{T}\Lambda$ as *improper, non-orthochronous* $\mathbf{L}_{\downarrow}^{(-)}$.

11.1 Parity

Space reflection must be represented in the Hilbert space by a unitary operator U_P. The quantized Dirac field should transform as

$$U_P \, \psi(x^0, \vec{x}) \, U_P^{-1} = \Lambda_P \, \psi(x^0, -\vec{x}) \tag{11.1}$$

where Λ_P is the 4×4 matrix representing space reflection.

A quick way to determine Λ_P is via the Dirac equation. We have

$$(i\gamma^\mu \partial_\mu - m) \, \psi(x) = 0 \Longrightarrow (i\gamma^\mu \partial_\mu - m) \, U_P \psi(x) U_P^{-1} = 0$$

or

$$(i\gamma^\mu \partial_\mu - m) \, \Lambda_P \, \psi(\tilde{x}) = 0 \Longrightarrow \Big(i\Lambda_P^{-1} \gamma^\mu \Lambda_P \partial_\mu - m \Big) \, \psi(\tilde{x}) = 0 \, ,$$

where $\tilde{x}^\mu = (x^0, \, -x^i)$. The space-reflected Dirac equation reads

$$\Big(i\gamma^\mu \tilde{\partial}_\mu - m \Big) \, \psi(\tilde{x}) = 0 \, .$$

Therefore, in view of the identity

$$\gamma^0 \, \gamma^\mu \, \gamma^0 = \begin{cases} \gamma^0 \\ \\ -\gamma^i \end{cases} = \gamma_\mu \, ,$$

we can conclude that, up to a phase, we must have $\Lambda_P \propto \gamma^0$.

Let's see how the above determination of the parity operator affects the plane wave expansion. One expects that the unitary operator $\mathbf{U}_P$ acting on one-particle states will give a one-particle state of the same spin with reversed three-momentum, namely[1]

$$\begin{aligned} U_P (a^{(s)}(\vec{k}))^\dagger |0\rangle &= \eta_a (a^{(s)}(-\vec{k}))^\dagger |0\rangle \\ U_P (b^{(s)}(\vec{k}))^\dagger |0\rangle &= \eta_b (b^{(s)}(-\vec{k}))^\dagger |0\rangle \end{aligned} \tag{11.2}$$

where η_a, η_b are proportionality constants. Since these constants should not influence normalization, they can only be phase factors

[1] We assume that the vacuum state obeys $U_P |0\rangle = |0\rangle$.

$\eta = e^{i\varphi}$. Substituting the plane wave expansion of $\psi(x)$ into (11.1) , we obtain

$$\sum_s \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E}} \left(\eta_a^* a^{(s)}(-\vec{k}) u^{(s)}(k) e^{-ik\cdot x} \,+\, \eta_b (b^{(s)}(-\vec{k}))^\dagger v^{(s)}(k) e^{ik\cdot x} \right) \,=$$

$$\Lambda_P \sum_s \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E}} \left(a^{(s)}(\vec{k}) u^{(s)}(k) e^{-ik\cdot\tilde{x}} \,+\, (b^{(s)}(\vec{k}))^\dagger v^{(s)}(k) e^{ik\cdot\tilde{x}} \right)$$

where again $\tilde{x}^\mu \,=\, (x^0,\, -x^i)$. At this point we note that

$$k\cdot\sigma \,=\, E \,+\, k^i\sigma_i \,=\, E + (-k^i)(-\sigma_i) \,=\, \tilde{k}\cdot\overline{\sigma}$$

and

$$k\cdot\overline{\sigma} \,=\, \tilde{k}\cdot\sigma$$

with $\tilde{k}^\mu = (k^0,\, -k^i)$. Thus, we may have

$$u^{(s)}(k) \,=\, \begin{pmatrix} \sqrt{k\cdot\sigma}\,\xi \\ \\ \sqrt{k\cdot\overline{\sigma}}\,\xi \end{pmatrix} \,=\, \begin{pmatrix} \sqrt{\tilde{k}\cdot\overline{\sigma}}\,\xi \\ \\ \sqrt{\tilde{k}\cdot\sigma}\,\xi \end{pmatrix} \,=\, \gamma^0\, u^{(s)}(\tilde{k}) \qquad (11.3)$$

and

$$v^{(s)}(k) \,=\, \begin{pmatrix} \sqrt{k\cdot\overline{\sigma}}\,\xi \\ \\ -\sqrt{k\cdot\sigma}\,\xi \end{pmatrix} \,=\, \begin{pmatrix} \sqrt{\tilde{k}\cdot\sigma}\,\xi \\ \\ -\sqrt{\tilde{k}\cdot\overline{\sigma}}\,\xi \end{pmatrix} \,=\, -\gamma^0\, v^{(s)}(\tilde{k})\,. \qquad (11.4)$$

Going back, we can write[2]

$$\sum_s \int \frac{d^3\tilde{k}}{(2\pi)^{3/2}\sqrt{2E}} \left(\eta_a^* a^{(s)}(\tilde{k}) \gamma^0 u^{(s)}(\tilde{k}) e^{-i\tilde{k}\cdot\tilde{x}} \,-\, \eta_b (b^{(s)}(\tilde{k}))^\dagger \gamma^0 v^{(s)}(\tilde{k}) e^{i\tilde{k}\cdot\tilde{x}} \right) \,=$$

$$\Lambda_P \sum_s \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E}} \left(a^{(s)}(k) u^{(s)}(k) e^{-ik\cdot\tilde{x}} \,+\, (b^{(s)}(k))^\dagger v^{(s)}(k) e^{ik\cdot\tilde{x}} \right)$$

from which, we can extract

$$\Lambda_P \,=\, \eta_a^*\,\gamma^0 \,=\, -\eta_b\,\gamma^0$$

[2]Note that $x\cdot k \,=\, \tilde{x}\cdot\tilde{k}$.

or

$$\eta_b = -\eta_a^*$$

and

$$\Lambda_P = \eta_a^* \gamma^0 . \tag{11.5}$$

The quantized Dirac spinor transformation under parity reads

$$U_P \psi(x^0, \vec{x}) U_P = \eta_a^* \gamma^0 \psi(x^0, -\vec{x}) . \tag{11.6}$$

The Pauli-conjugate field $\overline{\psi}(x)$ will transform under parity as

$$U_P \overline{\psi}(x) U_P^{-1} = \left(U_P \psi(x) U^{-1} \right)^\dagger \gamma^0 = \left(\eta_a^* \gamma^0 \psi(\tilde{x}) \right)^\dagger \gamma^0 = \eta_a \overline{\psi}(x^0, -\vec{x}) \gamma^0$$

Thus, we shall have

$$\overline{\psi} \gamma^\mu \partial_\mu \psi(x) \to \overline{\psi}(\tilde{x}) \gamma^0 \gamma^\mu \gamma^0 \partial_\mu \psi(\tilde{x}) = \overline{\psi}(\tilde{x}) \gamma^\mu \tilde{\partial}_\mu \psi(\tilde{x})$$

$$\overline{\psi}(x) \psi(x) \to \overline{\psi}(\tilde{x}) \gamma^0 \gamma^0 \psi(\tilde{x}) = \overline{\psi}(\tilde{x}) \psi(\tilde{x})$$

and the Dirac Action

$$\int d^4x \left(i \overline{\psi}(x) \gamma^\mu \partial_\mu \psi(x) - m \overline{\psi}(x) \psi(x) \right) = \int d^4\tilde{x} \left(i \overline{\psi}(\tilde{x}) \gamma^\mu \tilde{\partial}_\mu \psi(\tilde{x}) - m \overline{\psi}(\tilde{x}) \psi(\tilde{x}) \right)$$

is parity-invariant. In contrast, the bilinear $\overline{\psi}(x) \gamma_5 \psi(x)$, although invariant under continuous Lorentz transformations, transforms as

$$\overline{\psi}(\tilde{x}) \gamma^0 \gamma_5 \gamma^0 \psi(\tilde{x}) = -\overline{\psi}(\tilde{x}) \gamma_5 \psi(\tilde{x}) .$$

The quantity $\overline{\psi} \gamma_5 \psi$ is referred to as a *peudoscalar*. Similarly, the bilinear $\overline{\psi} \gamma^\mu \gamma_5 \psi$, a vector under continuous Lorentz transformations, is referred to as a *pseudovector or an axial vector*.

11.2 Time Reversal

As we have seen so far all symmetry transformations of the Action that we have considered correspond to unitary operators acting on the Hilbert space. Actually, this correspondence is more general and this "theorem" dictates that all symmetry transformations correspond either to a unitary or *antiunitary* operator. An antiunitary operator is one that is unitary and *antilinear* in the following sense

$$UU^\dagger \,=\, U^\dagger U \,=\, \mathbf{1}, \qquad Uc \,=\, c^*\, U \tag{11.7}$$

for any complex number c. The fact that time-reversal will necessarily turn out to be an antiunitary operator is already clear at the level of the simple Schroedinger theory. We have

$$\begin{array}{l} U_T|\Psi(t)\rangle \,=\, |\Psi(-t)\rangle \\ \\ [\,U_T,\, H\,] \,=\, 0 \end{array} \Longrightarrow \left\{ \begin{array}{l} H|\Psi(t)\rangle \,=\, i\frac{d}{dt}|\Psi(t)\rangle \\ \Downarrow \\ U_T H U_T^{-1}|\Psi(-t)\rangle \,=\, U_T i U_T^{-1}\frac{d}{dt}|\Psi(-t)\rangle \\ \Downarrow \\ H|\Psi(-t)\rangle \,=\, i\frac{d}{d(-t)}|\Psi(-t)\rangle \end{array} \right.$$

provided that $U_T i U_T^{-1} \,=\, -i$, i.e. U_T being antilinear.

Denoting with U_T the operator that corresponds to temporal inversion $x^\mu \to \tilde{x}^\mu \,=\, (-x^0,\, x^i)$, its action on a quantized Dirac spinor will be

$$U_T\, \psi(x)\, \mathbf{U}_T^{-1} \,=\, \Lambda_T\, \psi(\tilde{x})\,. \tag{11.8}$$

The most straightforward way to determine Λ_T is through the Dirac equation:

Assuming that U_T is antiunitary, we have

$$(\,-i\,(\gamma^\mu)^*\,\partial_\mu\,-m\,)\,U_T\psi(x)U_T^{-1} \,=\, 0 \Longrightarrow (\,-i\,(\gamma^\mu)^*\,\partial_\mu\,-m\,)\,\Lambda_T\psi(\tilde{x}) \,=\, 0\,,$$

where, here $\tilde{x}^\mu \,=\, (-x^0,\, x^i)$. Proceeding further we get

$$\left(i\Lambda_T^{-1}\gamma^\mu\Lambda_T\partial_\mu\,-m\right)\psi(\tilde{x}) \,=\, 0 \Longrightarrow \left(i\gamma^\mu\tilde{\partial}_\mu\,-m\right)\psi(\tilde{x}) \,=\, 0$$

provided that

$$\Lambda_T^{-1}\gamma^\mu\Lambda_T \,=\, \left\{ \begin{array}{l} -\gamma^0 \\ \\ \gamma^i \end{array} \right. \,=\, -\gamma_\mu\,. \tag{11.9}$$

It is easy to see that this is satisfied by the choice (up to a phase)

$$\Lambda_T \,=\, \eta_T\,\gamma^1\,\gamma^3\,. \tag{11.10}$$

Thus, the operator relation for time-reversal reads

$$U_T\,\psi(x)\,U_T^{-1} \;=\; \eta_T\,\gamma^1\gamma^3\,\psi(\tilde{x})\,. \tag{11.11}$$

Substituting in (11.11) the plane wave expansion of $\psi(x)$, we obtain

$$\int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E}}\Big(U_T a^{(1)}(k)U_T^{-1}(u^{(1)}(k))^* e^{ik\cdot x} \,+\, U_T a^{(2)}(k)U_T^{-1}(u^{(2)}(k))^* e^{ik\cdot x} \,+\,\ldots\Big)$$

$$= \eta_T \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E}}\Big(a^{(1)}(k)\,\gamma^1\gamma^3 u^{(1)}(k) e^{-ik\cdot\tilde{x}} \,+\, a^{(2)}(k)\,\gamma^1\gamma^3 u^{(2)}(k) e^{-ik\cdot\tilde{x}} \,+\,\cdots\Big)$$

$$= \eta_T \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E}}\Big(-ia^{(1)}(k)\,(u^{(2)}(k))^* e^{-ik\cdot\tilde{x}} \,+\, ia^{(2)}(k)\,(u^{(1)}(k))^* e^{-ik\cdot\tilde{x}} \,+\,\cdots\Big)$$

$$= \eta_T \int \frac{d^3\tilde{k}}{(2\pi)^{3/2}\sqrt{2E}}\Big(-ia^{(1)}(\tilde{k})\,(u^{(2)}(\tilde{k}))^* e^{i\tilde{k}\cdot x} \,+\, ia^{(2)}(\tilde{k})\,(u^{(1)}(\tilde{k}))^* e^{i\tilde{k}\cdot x} \,+\,\cdots\Big)$$

where $\tilde{k} \;=\; (k^0, -k^i)$. We have taken $\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and used that

$$\gamma^1\gamma^3 u^{(1)}(k) \;=\; -i(u^{(2)}(k))^*, \quad \gamma^1\gamma^3 u^{(2)}(k) \;=\; i(u^{(1)}(k))^*\,. \tag{11.12}$$

We can choose the phase $\eta_T \;=\; -i$. Then, from the above we read

$$\begin{aligned} U_T a^{(1)}(k)U_T^{-1} \;&=\; a^{(2)}(k) \\ U_T a^{(2)}(k)U_T^{-1} \;&=\; -a^{(1)}(k) \end{aligned} \tag{11.13}$$

11.3 Charge Conjugation

This is a discrete transformation that does not correspond to a spacetime discrete transformation. It can be defined in terms of a linear unitary operator that transforms a *"a"-particle* into a *"b"-particle (antiparticle)* of the same spin and momentum, namely

$$U_C\,a^{(s)}(k)U_C^{-1} \;=\; b^{(s)}(k), \qquad U_C\,b^{(s)}(k)U_C^{-1} \;=\; a^{(s)}(k)\,. \tag{11.14}$$

In terms of the quantized Dirac spinor $\psi(x)$ we have

$$U_C\,\psi(x)\,U_C^{-1}\,=\,\mathcal{C}(\psi^\dagger(x))^\perp \tag{11.15}$$

Substituting the plane wave expansion and using (11.14), we get

$$\sum_s\int\frac{d^3k}{(2\pi)^{3/2}\sqrt{2E}}\left(b^{(s)}(k)\,u^{(s)}(k)e^{-ik\cdot x}\,+\,(a^{(s)}(k))^\dagger v^{(s)}(k)e^{ik\cdot x}\right)\,=$$

$$\sum_s\int\frac{d^3k}{(2\pi)^{3/2}\sqrt{2E}}\left((a^{(s)}(k))^\dagger\,\mathcal{C}\,(u^{(s)}(k))^* e^{ik\cdot x}\,+\,b^{(s)}(k)\mathcal{C}(v^{(s)}(k))^* e^{-ik\cdot x}\right)$$

We may take

$$u^{(s)}(k)\,=\,\begin{pmatrix}\sqrt{\sigma\cdot k}\xi^{(s)}\\ \\ \sqrt{\overline{\sigma}\cdot k}\xi^{(s)}\end{pmatrix},\quad v^{(s)}(k)\,=\,\begin{pmatrix}\sqrt{\sigma\cdot k}(-i\sigma_2)(\xi^{(s)})^*\\ \\ -\sqrt{\overline{\sigma}\cdot k}(-i\sigma_2)(\xi^{(s)})^*\end{pmatrix}.$$

This implies

$$v^{(s)}(k)\,=\,\mathcal{C}(u^{(s)}(k))^*,\quad u^{(s)}(k)\,=\,\mathcal{C}(v^{(s)}(k))^*$$

and it is satisfied for

$$\mathcal{C}\,=\,-i\gamma^2\,. \tag{11.16}$$

Thus, finally, we have

$$U_C\,\psi(x)\,U_C^{-1}\,=\,-i\gamma^2\,\psi^*(x)\,=\,-i\gamma^2\gamma^0(\overline{\psi}(x))^\perp\,. \tag{11.17}$$