Chapter 13

QUANTUM ELECTRODYNAMICS (QED)

The formalism that has been developed can describe any number of quantized scalar, vector and spinor fields. Interactions among these fields can be introduce with appropriate Lorentz invariant terms in the Lagrangean. Nevertheless, almost always symmetry requirements beyond Lorentz invariance restrict severely these terms. For example, Quantum Electrodynamics (QED), the theory of electrons and photons, represented by a Dirac spinor ψ and a vector field $\mathcal{A}^\mu$, is invariant under a set of *gauge transformations*

$$\begin{aligned} \psi(x) \;\to\; \psi'(x) \;=\; e^{-ie\Lambda(x)}\,\psi(x) \\ \mathcal{A}^\mu(x) \;\to\; \mathcal{A}'^\mu(x) \;=\; \mathcal{A}^\mu(x) \,+\, \partial_\mu\Lambda(x) \end{aligned} \tag{13.1}$$

These transformations are *local*, i.e. their parameters depend on the spacetime point. Although the Lagrange density of pure electromagnetism

$$\mathcal{L}_{EM} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$$

respects this symmetry, the Dirac Lagrangean

$$\mathcal{L}_D = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

does not. In fact, we have

$$\overline{\psi}'(i\gamma^\mu\partial_\mu - m)\psi' \to \overline{\psi}\,(\,i\gamma^\mu\partial_\mu \,- m \,+\, e\gamma^\mu\partial_\mu\Lambda\,)\,\psi\,.$$

It is clear then that an extra term $e\overline{\psi}\gamma^\mu\psi\,\mathcal{A}^\mu$ is needed in order to cancel the Λ-term. In fact, the invariant Lagrange density should be

$$\mathcal{L}_{QED} \;=\; \overline{\psi}(x)\,(\,i\gamma^\mu\partial_\mu \;-e\,\gamma^\mu\mathcal{A}^\mu(x)\;-m)\,\psi(x) - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\,. \tag{13.2}$$

This is the Lagrangean of Quantum Electrodynamics (QED). The parameter e (electric charge of the electron) measures the strength of the interaction. Note that a *"photon mass"* term $\mu^2\mathcal{A}^\mu\mathcal{A}_\mu/2$ is forbidden by gauge invariance since it is not invariant under the transformations (13.1). Another way to look at the QED Lagrangean is that it is obtained from the sum of the Dirac and the pure electromagnetism Lagrangeans $\mathcal{L}_D + \mathcal{L}_{EM}$ by replacing the derivative in the Dirac Lagrangean with the so-called *covariant derivative*

$$\partial_\mu \Longrightarrow D_\mu \,\equiv\, \partial_\mu \,+\, ie\mathcal{A}_\mu\,. \tag{13.3}$$

The equations of motion derived fron (13.2) are

$$\begin{aligned} \Big(-\Box\, g_\mu^{\ \nu} \,+\, \partial_\mu\partial^\nu\Big)\,\mathcal{A}_\nu \;&=\; -e\overline{\psi}\gamma_\mu\psi \\ (i\gamma^\mu\partial_\mu \,- m)\,\psi \;&=\; e\mathcal{A}^\mu\gamma_\mu\psi \end{aligned} \tag{13.4}$$

The last one can also be written in terms of the covariant derivative as

$$(i\gamma^\mu D_\mu - m)\,\psi \;=\; 0\,. \tag{13.5}$$

It is straightforward to verify that the current is conserved, namely

$$\mathcal{J}^\mu \;=\; \overline{\psi}\gamma^\mu\psi \;\Longrightarrow\; \partial_\mu\mathcal{J}^\mu \;=\; 0\,. \tag{13.6}$$

The equations of motion (13.4) contain non-linear terms in their right-hand sides and cannot be solved in terms of plane wave ansätze. Nevertheless, for small values of the coupling constant, perturbation theory can be used, employing an iteration of the free field solutions.

Covariant derivatives have remarkable properties. First of all, it is easy to see that the covariant derivative of a spinor has exactly the same gauge transformation as the spinor itself, namely

$$\psi'(x) \,=\, e^{-ie\Lambda(x)}\psi(x) \Longrightarrow (D_\mu\psi)' \,=\, e^{-ie\Lambda(x)}\,\psi(x) \tag{13.7}$$

in contrast to the usual derivative. Another usefull property is the non-commutativity of covariant derivatives

$$[\,D_\mu,\, D_\nu\,] \,=\, ie\mathcal{F}_{\mu\nu}\,. \tag{13.8}$$

It is rather straightforward to formulate also a locally gauge invariant theory of a charged scalar field. Starting from the complex scalar Lagrangean and replacing the ordinary derivatives with covariant ones

$$(\partial^\mu\phi^\dagger)(\partial^\mu\phi) \Longrightarrow (D_\mu\phi)^\dagger(D^\mu\phi)\,.$$

The above kinetic term is invariant under the local gauge transformations

$$\begin{aligned} \phi(x) &\to \phi'(x) \,=\, e^{-ie\Lambda(x)}\phi(x) \\ \mathcal{A}_\mu(x) &\to \mathcal{A}'_\mu(x) \,=\, \mathcal{A}_\mu(x) + \partial_\mu\Lambda(x) \end{aligned} \tag{13.9}$$

Thus, we obtain the *Scalar QED* Lagrangean

$$\mathcal{L} \,=\, (D_\mu\phi)^\dagger(D^\mu\phi) - \frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - V\,, \tag{13.10}$$

where the *"potential"* V is a function of the gauge invariant combination of the scalar fields $\phi^\dagger\phi$. For reasons that can be clear at a much later point in the study of quantum field theory, the potential should not involve higher than quartic powers of the scalar field. Then, the most general gauge invariant choice is

$$V \,=\, \mu^2\,\phi^\dagger\phi \,+\, \lambda\,(\phi^\dagger\phi)^2\,.$$