Chapter 14

INTERACTING FIELDS

14.1 The Interaction Picture

The so-called *"Interaction Picture"* of time evolution is of central importance in the formulation of the perturbation theory for interacting fields. In this section we shall describe it in the simple framework of a Hermitean scalar field with a Lagrangean of the form

$$\mathcal{L} \,=\, \mathcal{L}_0 \,+\, \mathcal{L}_{int}\,,$$

where $\mathcal{L}_0$ is the familiar Klein-Gordon free part that we know how to solve in terms of creation-annihilation operators and $\mathcal{L}_{int}$ is an additional part describing a self interaction of the scalar field of the form of a single or more terms, each being a power of $\phi(x)$ at the same spacetime point (for example, $-\frac{\lambda}{3!}\phi^3(x)$). The full Hamiltonian is of the form[1]

$$\begin{aligned} H \,&=\, H_0 \,+\, H_{int} \\ &= \int d^3x\,\frac{1}{2}\left((\dot{\phi})^2 \,+\, (\vec{\nabla}\phi)^2 \,+\, m^2\phi^2\right) \,+\, \sum_{n\geq 3}\int d^3x\,\frac{\lambda_n}{n!}\phi^n(x)\,. \end{aligned} \tag{14.1}$$

[1]Since the full Hamiltonian is a constant of the motion, we shall assume that both H_0 and H_{int} are considered at the given time t_0.

The Heisenberg operator of the field $\phi(x)$ evolves in time with the full Hamiltonian as

$$\phi(t,\,\vec{x}) \;=\; e^{i(t-t_0)H}\,\phi(t_0,\,\vec{x})\,e^{-i(t-t_0)H}\,. \tag{14.2}$$

We are always free to take the field $\phi(t_0,\,\vec{x})$, *at a given instant of time* t_0 *(only)*, to coincide with the plane wave solution of the free theory $\phi_0(t_0,\,\vec{x})$, i.e.

$$\phi(t_0,\,\vec{x}) \;=\; \phi_0(t_0,\,\vec{x}) \;=\; \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}}\left(a(k)e^{-i\omega t_0+i\vec{k}\cdot\vec{x}} \;+\; a^\dagger(k)e^{i\omega t_0-i\vec{k}\cdot\vec{x}}\right)\,. \tag{14.3}$$

Then, the field $\phi(x)$ at all times will be

$$\phi(t,\,\vec{x}) \;=\; e^{i(t-t_0)H}\,\phi_0(t_0,\,\vec{x})\,e^{-i(t-t_0)H}\,. \tag{14.4}$$

If the interaction were absent the evolved field would be just

$$e^{i(t-t_0)H_0}\,\phi_0(t_0,\,\vec{x})\,e^{-i(t-t_0)H_0}\,.$$

We may then define this quantity as the *"interaction picture field"*

$$\phi_I(t,\,\vec{x}) \;\equiv\; e^{i(t-t_0)H_0}\,\phi_0(t_0,\,\vec{x})\,e^{-i(t-t_0)H_0}\,. \tag{14.5}$$

Since $\phi_I(t_0,\vec{x}) = \phi_0(t_0,\,\vec{x})$, it is clear that ϕ_I evolves with the free Hamiltonian, namely

$$\phi_I(t,\vec{x}) \;=\; e^{i(t-t_0)H_0}\phi_I(t_0,\vec{x})e^{-i(t-t_0)H_0}\,. \tag{14.6}$$

Inverting the above relations we may obtain the fully interacting Heisenberg field in terms of $\phi_I(x)$ at all times as

$$\phi(t,\,\vec{x}) \;=\; e^{i(t-t_0)H}\,e^{-i(t-t_0)H_0}\,\phi_I(t,\,\vec{x})\,e^{i(t-t_0)H_0}\,e^{-i(t-t_0)H}\,. \tag{14.7}$$

Next, we can introduce the evolution operator[2]

$$U(t,\,t_0) \;\equiv\; e^{i(t-t_0)H_0}\,e^{-i(t-t_0)H} \tag{14.8}$$

[2]This operator satisfies the following set of useful identities

$$U(t,\,t') \;=\; e^{iH_0(t-t_0)}e^{-i(t-t')H}e^{-iH_0(t'-t_0)}$$

and (for $t_1 \geq t_2 \geq t_3$)

$$U(t_1,\,t_2)U(t_2,\,t_3) \;=\; U(t_1,\,t_3)$$
$$U(t_1,\,t_3)U^\dagger(t_2,\,t_3) \;=\; U(t_1,\,t_2)\,.$$

and write (14.7) in terms of it as

$$\phi(t,\,\vec{x})\;=\;U^{\dagger}(t,\,t_0)\,\phi_I(t,\,\vec{x})\,U(t,\,t_0) \tag{14.9}$$

Note that states in the *interaction picture* evolve with (14.8). The evolution operator $U(t,\,t_0)$ satisfies a Schroedinger-like equation (obtained by simple differentiation)

$$\frac{dU(t,\,t_0)}{dt}\;=\;-ie^{i(t-t_0)H_0}H_{int}e^{i(t-t_0)H}\;=\;-iH_I\,U(t,\,t_0)\,, \tag{14.10}$$

where we have defined the *"interaction picture Hamiltonian"*

$$H_I\;=\;e^{i(t-t_0)H_0}H_{int}e^{-i(t-t_0)H_0}\,. \tag{14.11}$$

Note that for our simple polynomial choice of $H_{int}\;=\;\sum_{n\geq 3}\int\,d^3x\,\frac{\lambda_n}{n!}\phi^n(t_0,\,\vec{x})$, we get

$$H_I(t)\;=\;\sum_{n\geq 3}\int\,d^3x\,\frac{\lambda_n}{n!}e^{iH_0(t-t_0)}\phi^n(t_0,\,\vec{x})e^{-iH_0(t-t_0)}$$

$$=\;\sum_{n\geq 3}\int\,d^3x\,\frac{\lambda_n}{n!}e^{iH_0(t-t_0)}\phi_I^n(t_0,\,\vec{x})e^{-iH_0(t-t_0)}$$

$$=\;\sum_{n\geq 3}\int\,d^3x\,\frac{\lambda_n}{n!}\left(e^{iH_0(t-t_0)}\phi_I(t_0,\,\vec{x})e^{-iH_0(t-t_0)}\right)^n\;=\;\sum_{n\geq 3}\int\,d^3x\,\frac{\lambda_n}{n!}\phi_I^n(x)\,.$$

The operator differential equation (14.10) can be converted into an integral equation as follows

$$U(t,\,t_0)\;=\;\mathbf{1}\;-i\int_{t_0}^{t}dt'\,H_I(t')\,U(t',\,t_0)\,. \tag{14.12}$$

This can be converted into a series by successive iterations as

$$U(t,\,t_0)\;=\;\mathbf{1}\;-i\int_{t_0}^{t}dt'\,H_I(t')\;+\;(-i)^2\int_{t_0}^{t}dt_1\,\int_{t_0}^{t_1}dt_2\,H_I(t_1)H_I(t_2)$$

$$+\,(-i)^3\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}\,dt_2\,\int_{t_0}^{t_2}\,dt_3\,H_I(t_1)H_I(t_2)H_I(t_3)\;+\;\ldots\,. \tag{14.13}$$

Notice however that each of these terms can be written as a *time-ordered product*. For instance, the second order term is

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1)H_I(t_2) \;=$$

$$\frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1)H_I(t_2) \;+\; \frac{1}{2}\int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1\, H_I(t_2)H_I(t_1)$$

$$= \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, \Theta(t_1 - t_2)\, H_I(t_1)H_I(t_2) + \frac{1}{2}\int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_1\, \Theta(t_2 - t_1) H_I(t_2)H_I(t_1)$$

$$= \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\left(H_I(t_1)H_I(t_2)\right) \,.$$

Thus, finally, the evolution operator can be written as a *time-ordered exponential*

$$U(t,\, t_0) \;=\; T\left(e^{-i\int_{t_0}^{t} dt' H_I(t')}\right) \tag{14.14}$$

and

$$\phi(t,\, \vec{x}) \;=\; T\left(e^{-i\int_{t}^{t_0} dt' H_I(t')}\right) \phi_I(t,\, \vec{x})\, T\left(e^{-i\int_{t_0}^{t} dt' H_I(t')}\right) \,. \tag{14.15}$$

What about the vacuum state $|\Omega\rangle$ of the interacting theory? Let's consider the vacuum state of the free (unperturbed) theory $|0\rangle$ which is annihilated by H_0, namely $H_0|0\rangle = 0$. The evolution of this state in terms of the full Hamiltonian is

$$e^{-iHT}|0\rangle \;=\; \sum_n |n\rangle\, e^{-iE_nT}\, \langle n|0\rangle \;\approx\; e^{-iE_0T}\, \langle \Omega|0\rangle\, |\Omega\rangle\,, \tag{14.16}$$

where we have inserted a complete set $\{|n\rangle\}$ of eigenstates of H and the sum is approximated by the contribution of the ground state, dominant for large times $T \to \infty(1-i\epsilon)$. As a result, we may write

$$|\Omega\rangle \;=\; \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1}\, e^{-iHT}|0\rangle\,. \tag{14.17}$$

Shifting the infinite time T by a finite amount t_0 will not change (14.17), which becomes

$$|\Omega\rangle = \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(T+t_0)}|0\rangle$$

$$= \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(T+t_0)}e^{iH_0(T+t_0)}|0\rangle$$

$$= \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} \left(e^{-iH_0(T+t_0)}e^{iH(T+t_0)}\right)^{\dagger} |0\rangle$$

$$= \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} U^{\dagger}(-T, t_0)|0\rangle$$

For the second step we used the fact that $H_0|0\rangle = 0$. Thus, finally we obtain

$$|\Omega\rangle = \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} U(t_0,\, -T)|0\rangle\,. \tag{14.18}$$

If we make the replacement $T \to -T$ above and take the adjoint, we get

$$\langle\Omega| = \left(e^{iE_0(-T+t_0)}\langle 0|\Omega\rangle\right)^{-1} \langle 0|U(T,\, t_0)\,. \tag{14.19}$$

Taking the inner product of the last two expressions, we get

$$\langle\Omega|\Omega\rangle = 1 = \frac{e^{2iE_0T}}{|\langle 0|\Omega\rangle|^2}\langle 0|U(T,\, t_0)U(t_0,\, -T)|0\rangle$$

or

$$|\langle 0|\Omega\rangle|^2 = e^{2iE_0T}\langle 0|U(T,\, -T)|0\rangle\,.$$

Thus, any *exact-vacuum* expectation value can be written as a *free-vacuum* expectation value in the following way

$$\langle\Omega|\cdots|\Omega\rangle = \frac{e^{2iE_0T}}{|\langle 0|\Omega\rangle|^2}\langle 0|U(T,\, t_0)\cdots U(t_0,\, -T)|0\rangle$$

or

$$\langle\Omega|\cdots|\Omega\rangle = \frac{\langle 0|U(T,\, t_0)\cdots U(t_0,\, -T)|0\rangle}{\langle 0|U(T,\, -T)|0\rangle}\,. \tag{14.20}$$

Let's apply the formula (14.20) for a simple bilinear of field operators as it appears in the standard propagator, namely

$$\langle\Omega|\phi(x)\,\phi(y)|\Omega\rangle, \qquad (x^0 > y^0)\,.$$

According to (14.20) we have

$$(\langle 0|U(T,\,-T)|0\rangle)^{-1}\langle 0|U(T,\,t_0)U^\dagger(x_0,\,t_0)\phi_I(x)U(x_0,\,t_0)U^\dagger(y_0,\,t_0)\phi_I(y)U(y_0,\,t_0)U(t_0,\,-T)|0\rangle$$

$$= (\langle 0|U(T,\,-T)|0\rangle)^{-1}\langle 0|U(T,\,x_0)\phi_I(x)U(x_0,\,y_0)\phi_I(y)U(y_0,\,-T)|0\rangle$$

$$= (\langle 0|U(T,\,-T)|0\rangle)^{-1}\langle 0|T\left(U(T,\,x_0)\phi_I(x)U(x_0,\,y_0)\phi_I(y)U(y_0,\,-T)\right)|0\rangle$$

or

$$\langle\Omega|\phi(x)\,\phi(y)|\Omega\rangle \;=\; \frac{\langle 0|T\left(\phi_I(x)\phi_I(y)U(T,\,-T)\right)|0\rangle}{\langle 0|U(T,\,-T)\,)|0\rangle}\,.$$

The same is true for $y^0 > x^0$. Thus, we have in general

$$\langle\Omega|T\left(\phi(x)\phi(y)\right)|\Omega\rangle \;=\; \frac{\langle 0|T\left(\phi_I(x)\phi_I(y)\,e^{-i\int_{-T}^{T}dtH_I}\right)|0\rangle}{\langle 0|T\left(e^{-i\int_{-T}^{T}dtH_I}\right)|0\rangle}\,. \quad (14.21)$$

This formula can be generalized for any string of time-ordered field operators

$$\langle\Omega|T\left(\phi(x_1)\cdots\phi(x_n)\right)|\Omega\rangle \;=\; \frac{\langle 0|T\left(\phi_I(x_1)\cdots\phi_I(x_n)\,e^{-i\int_{-T}^{T}dt\,H_I}\right)|0\rangle}{\langle 0|T\left(e^{-i\int_{-T}^{T}dt\,H_I}\right)|0\rangle}\,. \quad (14.22)$$

Closing this section we should stress that in the interaction picture the field operators evolve in time with the *free* Hamiltonian and, since we can always assume that at a given time they are free fields, will be free at any time with their known expansion in terms of creation and annihilation operators. In contrast, the states evolve with the interaction picture evolution operator containing the interaction part of the Hamiltonian. Of course, observable quantities are inner products or matrix elements and are picture-independent.

14.2 The S-Matrix

Let's assume that our system is at the remote past in an initial state

$$|\psi(-\infty)\rangle = |i\rangle\,.$$

The system eventually evolves into a state

$$|\psi(+\infty)\rangle\,.$$

The probability that in the remote future the system is found in a specific final state $|f\rangle$, member of a complete set of states $\{|f\rangle\}$, will be

$$\mathcal{P}_{i\to f} = \left|\langle f|\psi(+\infty)\rangle\right|^2 = \left|\langle f|U(+\infty,\,-\infty)|i\rangle\right|^2\,.$$

The evolution operator between remote past and remote future is called *S-matrix*

$$S \equiv U(+\infty,-\infty)\,. \tag{14.23}$$

Thus, the above probability will be

$$\mathcal{P}_{i\to f} = |\langle f|S|i\rangle|^2 = |S_{fi}|^2\,. \tag{14.24}$$

Obviously S is unitary, since the evolution operator is unitary to begin with. Unitarity is expressed as

$$1 = \langle\psi(+\infty)|\psi(+\infty)\rangle = \langle\psi(+\infty)|\sum_f |f\rangle\langle f|\psi(+\infty)\rangle$$

$$= \sum_f \langle i|S^\dagger|f\rangle\langle f|S|i\rangle = \sum_f |S_{fi}|^2$$

or

$$\sum_f |S_{fi}|^2 = 1\,. \tag{14.25}$$

If the calculation of the S-matrix elements $\mathcal{S}_{fi} = \langle f|S|i\rangle$ is carried out in the *Interaction Picture*, then, the evolution operator that defines the S-matrix will be the *interaction picture* evolution operator (14.8). Using its expression as a time-ordered product, we can write the S-matrix as

$$S = T\left(e^{-i\int_{-\infty}^{+\infty} dt\, H_I(t)}\right) \tag{14.26}$$

and

$$S_{fi} \,=\, \langle f| T \left(e^{-i \int_{-\infty}^{+\infty} dt\, H_I(t)} \right) |i\rangle \,. \tag{14.27}$$

The state $|i\rangle$, a state at $t = -\infty$, is often characterized as an *"in state"*, while the final state $|f\rangle$, a state at $t = +\infty$, is said to be an *"out state"*. Thus, these states can be momentum eigenstated generated by the action of free creation operators on the unperturbed vacuum $|0\rangle$.

In most workable cases H_I is characterized by a small parameter. This can be shown explicitly by redefining $H_I \to \lambda H_I$. Then, we can write a perturbative expansion of the S-matrix as a power series in the coupling constant λ. It reads

$$S \,=\, \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} S^{(n)} \,, \tag{14.28}$$

with

$$S^{(n)} \,=\, (-i)^n \int dt_1 \int dt_2 \ldots \int dt_n \, T\left(H_I(t_1) H_I(t_2) \ldots H_I(t_n) \right)$$

or, in terms of the Hamiltonian density

$$S^{(n)} \,=\, (-i)^n \int d^4x_1 \int d^4x_2 \ldots \int d^4x_n \, T\left(\mathcal{H}_I(x_1) \mathcal{H}_I(x_2) \ldots \mathcal{H}_I(x_n) \right) \,. \tag{14.29}$$

Since $\mathcal{L}_{int}$ is usually a monomial or polynomial of powers of the field operator, the computation of $S^{(n)}$ is reduced to computing time-ordered products of $\phi_I(x)$ operators

$$\langle f| T \left(\phi_I(x_1) \phi_I(x_2) \ldots \phi_I(x_n) \right) |i\rangle \,.$$

It is usefull to separate the trivial non-interaction part in S and put all interaction in the so-called *T-matrix*, defined as

$$S \,=\, \mathbf{1} \,+\, iT \,. \tag{14.30}$$

Taking the asymptotic states to be momentum eigenstates[3] we may write

$$\langle p_f^{(1)}, \, p_f^{(2)}, \ldots |iT| p_i^{(1)}, \, p_i^{(2)}, \ldots \rangle \,=\, \delta\left(\sum p_i - \sum p_f \right) \, i\mathcal{M} \,. \tag{14.31}$$

The function $\mathcal{M}$ is called *the invariant amplitude* and it is directly related to the scattering cross section.

[3]The external momenta $p_i^{(n)}$, $p_f^{(n)}$ are all *on mass-shell*.

14.3 Wick's theorem

Any (free) field operator can be written as a sum of a *positive frequency* and a *negative frequency* part

$$\phi(x) \,=\, \phi_+(x) \,+\, \phi_-(x)\,. \tag{14.32}$$

For the case of the free scalar field

$$\phi_+(x) \,=\, \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}}\, a(k)\, e^{-ik\cdot x}, \quad \phi_-(x) \,=\, \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}}\, a^\dagger(k)\, e^{ik\cdot x}\,. \tag{14.33}$$

Introducing this separation into the time-ordered product, we get for $t > t'$

$$T\left(\phi(x)\phi(x')\right) \,=\, \phi_+(x)\phi_+(x')+\phi_+(x)\phi_-(x')+\phi_-(x)\phi_+(x')+\phi_-(x)\phi_-(x') \,=$$

$$\phi_+(x)\phi_+(x')+\phi_-(x')\phi_+(x)+[\phi_+(x),\,\phi_-(x')]+\phi_-(x)\phi_+(x')+\phi_-(x)\phi_-(x')$$

or

$$T\left(\phi(x)\phi(x')\right) \,=\, N\left(\phi(x)\phi(x')\right) \,+\, [\phi_+(x),\,\phi_-(x')]\;, \tag{14.34}$$

where N stands for the *normal ordered product*. Taking the vacuum expectation value of both sides, we get

$$\langle 0|T\left(\phi(x)\phi(x')\right)|0\rangle \,=\, \langle 0|N\left(\phi(x)\phi(x')\right)|0\rangle \,+\, \langle 0|\,[\phi_+(x),\,\phi_-(x')]\,|0\rangle$$

or

$$\langle 0|T\left(\phi(x)\phi(x')\right)|0\rangle \,=\, \langle 0|\,[\phi_+(x),\,\phi_-(x')]\,|0\rangle \tag{14.35}$$

or, introducing the *Feynman propagator*

$$i\Delta_F(x-x') \,=\, \langle 0|\,[\phi_+(x),\,\phi_-(x')]\,|0\rangle\,. \tag{14.36}$$

Note however that the vacuum expectation value in the right hand side is superfluous and the commutator is already a c-number

$$[\phi_+(x),\,\phi_-(x')] \,=\, \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \int \frac{d^3k'}{\sqrt{2\omega'(2\pi)^3}} \left[\,a(k),\,a^\dagger(k')\,\right]\, e^{-ik\cdot x+ik'\cdot x'}$$

$$= \int \frac{d^3k}{2\omega(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')} .$$

Therefore, (14.34) can be written as

$$T\left(\phi(x)\phi(x')\right) \;=\; N\left(\phi(x)\phi(x')\right) \;+\; i\Delta_F(x-x') . \qquad (14.37)$$

This is the simplest case of *Wick's theorem*. The general statement is that *any time-ordered product can be converted into a normal-ordered product and a c-number propagator.*

In order to state generally Wick's theorem we may introduce the concept of *contraction* of operators. For two field operators, their contraction simply corresponds to their propagator

$$\underbrace{\phi(x)\phi(x')} \;=\; i\Delta_F(x-x') . \qquad (14.38)$$

Wick's Theorem states that any time-ordered product can be reduced to normal-ordered products and contractions according to

$$T\left(ABC\ldots\right) \;=\; N\left(ABC\ldots\right) + N\left(\underbrace{AB}C\ldots\right) + N\left(A\underbrace{BC}\ldots\right) + N\left(\underbrace{ABC}\ldots\right) ,$$

where

$$N\left(\underbrace{AB}C\ldots\right) \;=\; \underbrace{AB}\, N\left(C\ldots\right) .$$

For example, in the case that we have a string of four operators, we write symbolically

$$T(abcd) \;=\; N(abcd) +$$

$$\underbrace{ab}\,N(cd) + \underbrace{ac}\,N(bd) + \underbrace{ad}\,N(bc) + \underbrace{bc}\,N(ad) + \underbrace{bd}\,N(ac) + \underbrace{cd}\,N(ab)$$
$$+\ \underbrace{ab}\,\underbrace{cd} + \underbrace{ac}\,\underbrace{bd} + \underbrace{ad}\,\underbrace{bc} + \underbrace{bc}\,\underbrace{ad} + \underbrace{bd}\,\underbrace{ac} + \underbrace{cd}\,\underbrace{ab} .$$

14.4 An Example: ϕ^3-theory

Consider the scalar Klein-Gordon field with a cubic self-interaction term in the Lagrangean

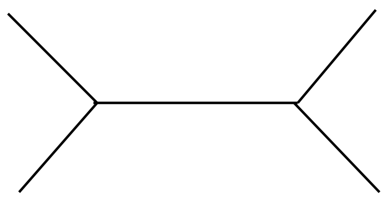
$$\mathcal{L}_{int} \;=\; -\frac{\lambda}{3!}\phi^3(x) . \qquad (14.39)$$

This corresponds to an interaction-picture S-matrix

$$S \;=\; T\left(e^{-i\frac{\lambda}{3!}\int d^4x\,\phi^3(x)}\right)\,. \tag{14.40}$$

If we are interested in a non-trivial *two-particle scattering* process

$$\phi + \phi \to \phi + \phi$$



we must have at least two interaction vertices. Thus, we should be interested in the $O(\lambda^2)$ term, i.e.

$$S_2 \;=\; \frac{1}{2!}\left(-\frac{i\lambda}{3!}\right)^2\int d^4x\int d^4x'\,T\left(\,\phi^3(x)\,\phi^3(x')\,\right)\,. \tag{14.41}$$

Applying Wick's theorem, we obtain

$$T\left(\,\phi^3(x)\,\phi^3(x')\,\right) \;=\; N\left(\,\phi^3(x)\,\phi^3(x')\,\right) + 9\,\underbrace{\phi(x)\phi(x')}\,N\left(\phi^2(x)\phi^2(x')\right) \;+$$

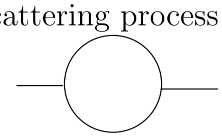
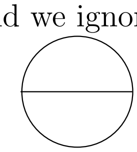
$$36\,\underbrace{\phi(x)\phi(x')}\,\underbrace{\phi(x)\phi(x')}\;N\left(\phi(x)\phi(x')\right) + 36\,\underbrace{\phi(x)\phi(x')}\,\underbrace{\phi(x)\phi(x')}\,\underbrace{\phi(x)\phi(x')}\,.$$

We ignored contractions at the same point, which have no physical meaning and could be made to cancel out. Thus, we have

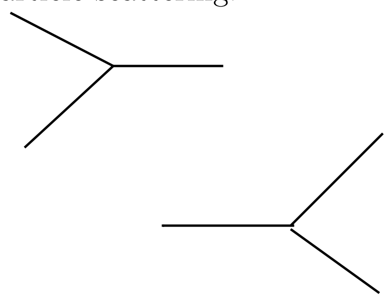
$$S_2 \;=\; \frac{1}{2!}\left(-\frac{i\lambda}{3!}\right)^2\int d^4x\int d^4x'\,\Big\{\,N\left(\,\phi^3(x)\,\phi^3(x')\,\right) \;+$$

$$9i\Delta_F(x-x')\,N\left(\phi^2(x)\phi^2(x')\right) \;+\; 36i^2\Delta_F^2(x-x')\,N\left(\phi(x)\phi(x')\right) \;+\; 36i^3\Delta_F^3(x-x')\,\Big\}\,.$$

The last two terms involve one and two closed loops and have either only two or no external legs. Therefore, they cannot contribute to the two-particle scattering process and we ignore them.

Similarly, the first term is a product of two independent processes that proceed independently and, thus, does not contribute either to the two-particle scattering.



Thus, we have only the second term which is

$$S_2 \,=\, \frac{9i}{2!}\left(-\frac{i\lambda}{3!}\right)^2 \int d^4x \int d^4x'\,\Delta_F(x-x')\,N\left(\phi^2(x)\phi^2(x')\right)\,.$$

Ignoring terms that do not contribute to the absorption or annihilation of two particles, we obtain

$$S_2 \,=\, -i\frac{\lambda^2}{4}\int d^4x \int d^4x'\,\Delta_F(x-x')\,\Big(\,\phi_-^2(x)\phi_+^2(x') \,+\, 2\phi_-(x)\phi_-(x')\phi_+(x)\phi_+(x')\Big)\,. \tag{14.42}$$

For the two particle process that we are interested in the initial and final states will be two-particle states of given momenta, namely

$$\begin{aligned} |k_1,\, k_2\rangle \,&=\, 2\sqrt{\omega_1\omega_2}\,a^\dagger(k_1)a^\dagger(k_2)|0\rangle \\ |f\rangle \,=\, |k_1',\, k_2'\rangle \,&=\, 2\sqrt{\omega_1'\omega_2'}\,a^\dagger(k_1')a^\dagger(k_2')|0\rangle \end{aligned} \tag{14.43}$$

The S-matrix element S_{fi} between these states is

$$-\frac{i\lambda^2}{4}\int d^4x \int d^4x'\Delta_F(x-x')\langle k_1',\, k_2'|\,\Big(\,\phi_-^2(x')\phi_+^2(x) \,+\, 2\phi_-(x)\phi_-(x')\phi_+(x)\phi_+(x')\Big)\,|k_1,\, k_2\rangle\,. \tag{14.44}$$

The fields acting on the states give

$$\phi_+(x')|k_1,\, k_2\rangle \,=\, \int \frac{d^3q}{\sqrt{2\omega(q)(2\pi)^3}}\,e^{-iq\cdot x'}a(q)|k_1,\, k_2\rangle \,=$$

$$\frac{1}{\sqrt{(2\pi)^{3/2}}}\,\Big(\sqrt{2\omega_2}\,e^{-ik_1\cdot x'}a^\dagger(k_2) \,+\, \sqrt{2\omega_1}\,e^{-ik_2\cdot x'}a^\dagger(k_1)\Big)\,|0\rangle\,.$$

Furthermore, we have

$$\phi_+(x)\phi_+(x')|k_1,\,k_2\rangle \;=\; \tfrac{1}{(2\pi)^3}\left(e^{-ik_2\cdot x'-ik_1\cdot x}\;+\;e^{-ik_1\cdot x'-ik_2\cdot x}\right)|0\rangle$$

$$\phi_+^2(x)|k_1,\,k_2\rangle \;=\; \tfrac{2}{(2\pi)^3}\,e^{-i(k_1+k_2)\cdot x}\,|0\rangle$$

The conjugate states are

$$\langle k_1'k_2'|\phi_-(x)\phi_-(x') \;=\; \tfrac{1}{(2\pi)^3}\langle 0|\left(e^{ik_2'\cdot x'+ik_1'\cdot x}\;+\;e^{ik_1'\cdot x'+ik_2'\cdot x}\right)$$

$$\langle k_1'k_2'|\phi_-^2(x') \;=\; \tfrac{2}{(2\pi)^3}\langle 0|e^{i(k_1'+k_2')\cdot x}$$

The corresponding matrix elements are

$$\langle k_1'k_2'|\phi_-(x)\phi_-(x')\phi_+(x)\phi_+(x')|k_1k_2\rangle \;=$$

$$\tfrac{1}{(2\pi)^6}\left(e^{ik_2'\cdot x'+ik_1'\cdot x}\;+\;e^{ik_1'\cdot x'+ik_2'\cdot x}\right)\left(e^{-ik_2\cdot x-ik_1\cdot x'}+e^{-ik_1\cdot x-ik_2x'}\right)$$

$$\langle k_1'k_2'|\phi_-^2(x')\phi_+^2(x)|k_1k_2\rangle \;=\; \tfrac{4}{(2\pi)^6}e^{i(k_1'+k_2')\cdot x'-i(k_1+k_2)\cdot x}$$

Thus, we have

$$\langle k_1',\,k_2'|\left(\phi_-^2(x')\phi_+^2(x)\;+\;2\phi_-(x)\phi_-(x')\phi_+(x)\phi_+(x')\right)|k_1,\,k_2\rangle \;=$$

$$\frac{4}{(2\pi)^6}\left(e^{i(k_1'+k_2')\cdot x-i(k_1+k_2)\cdot x}\;+\;e^{i(k_2'-k_1)\cdot x'+i(k_1'-k_2)\cdot x}\;+\;e^{i(k_1'-k_1)\cdot x'+i(k_2'-k_2)\cdot x}\right).$$

The resulting S-matrix element is

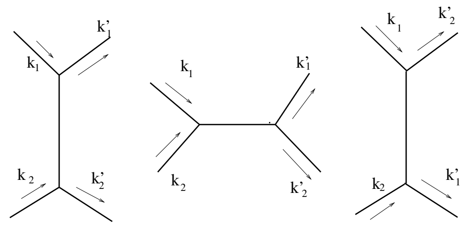
$$S_{fi} \;=\; -i\frac{\lambda^2}{(2\pi)^6}\int d^4x\int d^4x'\,\Delta_F(x-x')\,\left(e^{-i(k_1+k_2)\cdot(x-x')}\right.$$
$$\left.+\,e^{i(k_1'-k_2)\cdot(x-x')}\;+\;e^{i(k_2'-k_2)\cdot(x-x')}\right)e^{i(k_1'+k_2'-k_1-k_2)\cdot x'} \qquad (14.45)$$

or

$$S_{fi} \;=\; -i\lambda^2\,\delta(k_1+k_2-k_1'-k_2')\,\left(\tilde{\Delta}_F(k_1+k_2)\;+\;\tilde{\Delta}_F(k_1'-k_2)\;+\;\tilde{\Delta}_F(k_2'-k_2)\right), \qquad (14.46)$$

where

$$\tilde{\Delta}_F(q) \;=\; \int\frac{d^4x}{(2\pi)^2}\,\Delta_F(x)\,e^{iq\cdot x} \;=\; \frac{1}{q^2-m^2+i\epsilon}\,.$$



The delta function encodes four-momentum conservation. In addition, we observe that this amplitude possesses *crossing symmetry*, i.e. invariance under internal line changes with external legs fixed. We can introduce the *Mandelstam variables*

$$s = (k_1 + k_2)^2,\ t = (k_1' - k_1)^2,\ u = (k_1' - k_2)^2 \tag{14.47}$$

in terms of which the previously computed scattering amplitude has the form

$$\tilde{\Delta}_F(s) + \tilde{\Delta}_F(t) + \tilde{\Delta}_F(u)\,. \tag{14.48}$$

Note the relation

$$s + t + u = 4m^2\,. \tag{14.49}$$

14.5 Feynman Rules

As in the case of the ϕ^3 example that was worked out the different terms in the S-matrix elements or the correlation functions $\langle 0|T(\phi(x_1)\ldots\phi(x_n))|0\rangle$ have a useful graphical representation in terms of the so-called *Feynman Diagrams*. It is also possible to associate a precise value to each Feynman diagram that represents a term in a given correlation function, provided we adhere to a set of *Feynman Rules*. These are a set of rules for the calculation of the invariant matrix element $\mathcal{M}$ of an arbitrary scattering process, directly related to the scattering cross section.

In the case of $-\frac{\lambda}{3!}\phi^3$ interaction of the Hermitean scalar field these rules are:

1. For each line carrying momentum p, we associate a propagator $i\tilde{\Delta}_F(p) = \frac{i}{p^2-m^2+i\epsilon}$.
2. For each interaction vertex we associate $-i\lambda$
3. We impose momentum conservation at each vertex.
4. We integrate over each undetermined momentum as $\int d^4p$.

5. We multiply by the *symmetry factor*.

Applying these rules for the diagrams of the previous section we obtain

$$\int d^4q\,(-i\lambda)\delta(q-k_2+k_2')\tfrac{i}{q^2-m^2+i\epsilon}\delta(q-k_1'+k_1)(-i\lambda)+$$

$$\int d^4q\,(-i\lambda)\delta(q-k_1-k_2)\tfrac{i}{q^2-m^2+i\epsilon}\delta(q-k_1'-k_2')(-i\lambda)+$$

$$\int d^4q\,(-i\lambda)\delta(q-k_2+k_1')\tfrac{i}{q^2-m^2+i\epsilon}\delta(q+k_1-k_2')(-i\lambda)$$

$$= -i\lambda^2\delta(k_1+k_2-k_1'-k_2')\left(\frac{1}{(k_2-k_2')^2-m^2+i\epsilon}+\frac{1}{(k_1+k_2)^2-m^2+i\epsilon}\right.$$
$$\left.+\frac{1}{(k_2-k_1')^2-m^2+i\epsilon}\right)$$
$$= -i\lambda^2\delta(k_1+k_2-k_1'-k_2')\left(\frac{1}{t-m^2+i\epsilon}+\frac{1}{s-m^2+i\epsilon}+\frac{1}{u-m^2+i\epsilon}\right).$$

The symmetry factor is calculated to be $3\times3\times2\times2/(3!)^2=1$.

The Feynman Rules for QED are:

1. For each fermion line carrying momentum p, we associate a propagator

$$i\tilde{S}_F(p) \;=\; \frac{i}{\gamma\cdot p - m + i\epsilon} \;=\; i\frac{\gamma\cdot p + m}{p^2 - m^2 + i\epsilon}\,.$$

2. For each photon line carrying momentum p, we associate a propagator[4]

$$i\tilde{\Delta}_F^{\mu\nu}(p) \;=\; -\frac{ig^{\mu\nu}}{p^2 + i\epsilon}\,.$$

3. For each interaction vertex we associate $-ie\gamma^\mu$.

4. For each external *fermion* line $\underbrace{\psi|p,s;+\rangle}$, $\underbrace{\langle p,s;+|\overline{\psi}}$ we insert a factor $u^{(s)}(p)$, $\overline{u}^{(s)}(p)$.

5. For each external *antifermion* line $\underbrace{\langle p,s;-|\psi}$, $\underbrace{\overline{\psi}|p,s;-\rangle}$ we insert a factor $v^{(s)}(p)$, $\overline{v}^{(s)}(p)$.

6. For each external photon line $\underbrace{\mathcal{A}_\mu|p,\,\alpha\rangle}$, $\underbrace{\langle p,\,\alpha|\mathcal{A}_\mu}$ we insert a factor $\epsilon_\mu^{(\alpha)}(p)$, $(\epsilon_\mu^{(\alpha)}(p))^*$.

7. We impose momentum conservation at each vertex.

8. We integrate over each undetermined momentum as $\int d^4p$.

9. For each fermion loop we insert a factor of (-1).

10. We multiply by the *symmetry factor*.

14.6 The LSZ Reduction Formula

Consider a generic S-matrix element

$$\langle p_f^{1)}, p_f^{(2)}, \ldots; T_f|p_i^{(1)}, p_i^{(2)}, \ldots; T_i\rangle\,, \tag{14.50}$$

where it is understood that $T_f \to +\infty$ and $T_i \to -\infty$. For definiteness and simplicity we consider only one type of particles corresponding to the excitations of a real scalar field $\phi(x)$. Since we

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[4]This is only in the Feynman gauge ($\alpha = 1$). For general α, the propagator $i\tilde{\Delta}_F^{\mu\nu}(p)$ is

$$-\frac{i}{p^2 + i\epsilon}\left(g^{\mu\nu} - (1-\alpha)\frac{p^\mu p^\nu}{p^2}\right)$$

expect that in the circumstances of a scattering experiment the initial and final states will be to a good approximation eigenstates of the free-theory, we may assume that

$$T_i \to -\infty, \;\; \phi(x) \,=\, Z^{1/2}\,\phi_{in}(x)$$
$$T_f \to +\infty, \;\; \phi(x) \,=\, Z^{1/2}\,\phi_{out}(x) \tag{14.51}$$

where, $\phi_{in}(x)$ and $\phi_{out}(x)$ are free fields. Z stands fora normalization constant.

Going back to the expansion of a free scalar field $\phi(x)$, we can prove the following identities

$$(2\omega(2\pi)^3)^{1/2}\,a(k) \,=\, i\int d^3x\,e^{ik\cdot x}\,(\,\partial_0\phi(x)\,-i\omega\phi(x)\,)$$
$$(2\omega(2\pi)^3)^{1/2}\,a^\dagger(k) \,=\, -\,i\int d^3x\,e^{-ik\cdot x}\,(\,\partial_0\phi(x)\,+i\omega\phi(x)\,) \tag{14.52}$$

The identities (14.52) cam be shown in a straightforward fashion by substituting (5.2) in them.

Thus, in infinite past and future we may have

$$T_i \to -\infty \qquad -iZ^{-1/2}\int d^3x\,e^{-ik\cdot x}\,(\,\partial_0\phi(x)\,+i\omega\phi(x)\,) \,=\, (2\omega(2\pi)^3)^{1/2}\,a^\dagger_{in}(k)$$

$$T_f \to +\infty \qquad -\,i\,Z^{-1/2}\int d^3x\,e^{-ik\cdot x}\,(\,\partial_0\phi(x)\,+i\omega\phi(x)\,) \,=\, (2\omega(2\pi)^3)^{1/2}\,a^\dagger_{out}(k)$$

Note however that for any funcion $f(x)$ the following is true for $T_f \to +\infty$ and $T_i \to -\infty$

$$\int d^3x\,f(\vec{x},\,T_f) - \int d^3x\,f(\vec{x},\,T_i) \,=\, \int d^4x\,\partial_0 f(x)\,.$$

Using that, we obtain

$$(2\omega(2\pi)^3)^{1/2}\,\left(a^\dagger_{out}(k)\,-a^\dagger_{in}(k)\right) \,=\, -iZ^{-1/2}\int d^4x\,\partial_0\left(e^{-ik\cdot x}\,(\,\partial_0\phi(x)\,+i\omega\phi(x)\,)\right)$$

$$=\, -iZ^{-1/2}\int d^4x\,\left(\omega^2 e^{-ik\cdot x}\phi(x)\,+\,e^{-ik\cdot x}\partial_0^2\phi(x)\right)$$

$$=\, -iZ^{-1/2}\int d^4x\,\left(\phi(x)\left(-\nabla^2+m^2\right)e^{-ik\cdot x}\,+\,e^{-ik\cdot x}\partial_0^2\phi(x)\right)$$

or

$$(2\omega(2\pi)^3)^{1/2}\,\left(a^\dagger_{out}(k)\,-a^\dagger_{in}(k)\right)\,=\,-iZ^{-1/2}\int d^4x\,e^{-ik\cdot x}\left(\Box\,+\,m^2\right)\phi(x)\,. \tag{14.53}$$

considering again the S-matrix element, we have

$$\langle\,p_f^{(1)},\,\ldots;\,T_f|\,p_i^{(1)},\,\ldots;\,T_i\rangle\,=$$

$$\left(2\omega(p_i^{(1)})\right)^{1/2}\langle\,p_f^{(1)},\,\ldots;\,T_f|a^\dagger_{in}(p_i^{(1)})|\,p_i^{(2)},\,\ldots;\,T_i\rangle\;.$$

On the other hand, if we assume that the all $p_i^{(n)}\neq\,p_f^{(m)}$,

$$\langle\,p_f^{(1)},\,p_f^{(2)},\,\ldots;\,T_f|a^\dagger_{out}(p_i^{(1)})\,=\,0\,.$$

Thus, we may write

$$\langle\,p_f^{(1)},\,\ldots;\,T_f|\,p_i^{(1)},\,\ldots;\,T_i\rangle\,=$$

$$\left(2\omega(p_i^{(1)})\right)^{1/2}\langle\,p_f^{(1)},\,\ldots;\,T_f|\left(a^\dagger_{in}(p_i^{(1)})\,-a^\dagger_{out}(p_i^{(1)})\right)|\,p_i^{(2)},\,\ldots;\,T_i\rangle$$

and

$$\langle\,p_f^{(1)},\,\ldots;\,T_f|\,p_i^{(1)},\,\ldots;\,T_i\rangle\,=$$

$$iZ^{-1/2}(2\pi)^{-3/2}\int d^4x\,e^{-ip_i^{(1)}\cdot x}\left(\Box+m^2\right)\langle\,p_f^{(1)},\,\ldots;\,T_f|\phi(x)|\,p_i^{(2)},\,\ldots;\,T_i\rangle\,. \tag{14.54}$$

We may continue to *"subtract"* particles from the "in" and "out" states as follows:

$$\langle\,p_f^{(1)},\,\ldots;\,T_f|\phi(x)\,=\,\langle\,p_f^{(2)},\,\ldots;\,T_f|a_{out}(p_f^{(1)})\phi(x)\,=\,\langle\,p_f^{(2)},\,\ldots;\,T_f|T\left(a_{out}(p_f^{(1)})\phi(x)\right)\,.$$

Thus, we obtain

$$\langle\,p_f^{(1)},\,\ldots;\,T_f|\,p_i^{(1)},\,\ldots;\,T_i\rangle\,=$$

$$\left(iZ^{-1/2}(2\pi)^{-3/2}\right)^2\int d^4x\,e^{-ip_i^{(1)}\cdot x}\left(\Box_x+m^2\right)\int d^4y\,e^{-ip_f^{(1)}\cdot y}\left(\Box_y+m^2\right)$$

$$\langle\,p_f^{(2)},\,\ldots;\,T_f|T\left(\phi(y)\phi(x)\right)|\,p_i^{(2)},\,\ldots;\,T_i\rangle\,. \tag{14.55}$$

Repeating this process until we deplete the "in" and "out" states from any particles, we end up with

$$\langle\, p_f^{(1)},\, \dots;\, T_f |\, p_i^{(1)},\, \dots;\, T_i\rangle \;=\; \left(iZ^{-1/2}(2\pi)^{-3/2}\right)^{N_i+N_f}$$

$$\Pi_{j=1}^{N_i}\,\Pi_{k=1}^{N_f}\int d^4x_j \int d^4x'_k\, \left(\Box_j + m^2\right)\left(\Box'_k + m^2\right)$$

$$\langle\,\Omega| T\left(\phi(x_1)\phi(x_2)\dots\phi(x_{N_f})\,\phi(x'_1)\phi(x'_2)\cdots\phi(x_{N_i})\right)|\Omega\rangle \qquad (14.56)$$

This is the so-called *Lehman-Symanzik-Zimermann Reduction Formula*. It reduces any S-matrix element into the vacuum expectation value of a time-ordered product of field operators. The latter is, of course, picture-independent but can be related to an interaction picture expectation value through formula (14.22).

The vacuum expectation values

$$G_N(x_1,\, x_2,\, \dots\, x_N) \;=\; \langle\,\Omega| T\left(\phi(x_1)\phi(x_2)\dots\phi(x_N)\right)|\Omega\rangle$$

are the so-called *N-point functions*. It can be shown that they satisfy a system of coupled Green's function equations. They are often called *"causal"* Green's functions to be distinguished from the *"connected"* N-point functions from which all disconnected pieces have been subtracted. For example,

$$G_c^{(4)}(x_1,x_2,x_3,x_4) \;=\; G_4(x_1,x_2,x_3,x_4) \;-G_3(x_1,x_2,x_3)G_1(x_4)$$

$$-G_3(x_1,x_2,x_4)G_1(x_3) - G_3(x_1,x_3,x_4)G_1(x_2) - G_3(x_2,x_3,x_4)G_1(x_1)$$

$$-G_2(x_1,x_2)G_2(x_3,x_4) - G_2(x_1,x_3)G_2(x_2,x_4) - G_2(x_1,x_4)G_2(x_2,x_3)\,.$$

note that $G_1(x) = \langle\Omega|\phi(x)|\Omega\rangle$ is not necessarily zero as in free field theory.