Chapter 15

APPLICATIONS

15.1 The concept of scattering cross section

Consider a scattering experiment in which a beam of particles of mass m_1 and velocity v_1 is scattered by a target of particles of mass m_2. The number os scattering events per unit volume per unit time will be proportional to the number of incoming particles per unit volume times their velocity $v_1 n_1$ and to the density of the target (number of target particles per unit volume n_2)

$$\frac{dN_{sc}}{dV\,dt} = \sigma\, n_1\, v_1\, n_2\,, \tag{15.1}$$

where the proportionality constant σ is the *scattering cross section* of the collision events at hand. It is clear from (15.1) that the scattering cross section σ has dimensions of $(length)^2$. The cross section is assumed to be by definition Lorentz invariant and so is the number of scattering events per unit time per unit volume in the right-hand side of (15.1). Thus, the combination $n_1 n_2 v_1$, considered above in the rest frame of the target, must be Lorentz invariant as well. It has been calculated that the corresponding Lorentz invariant expression is $n_1 n_2 \sqrt{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2}$.

The number of scattering events divided by the numbers of incoming and target particles $N_{sc}/(n_1 V)(n_2 V)$ is just the scattering propability $\mathcal{P} = |S_{fi}|^2$. Thus, we may write

$$\sigma = |S_{fi}|^2 \frac{V}{vT} \tag{15.2}$$

where $v = \sqrt{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2}$ the *"incoming flux"*. Going a step further we write

$$|S_{fi}|^2 = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle} = \left((2E_1)(2E_2)\tilde{\delta}^{(3)}(0)\tilde{\delta}^{(3)}(0)\right)^{-1} \Pi_{n=1}^{N_f} \left(2E'_n\tilde{\delta}^{(3)}(0)\right)^{-1}$$

$$\tilde{\delta}^{(4)}(0) \sum_{n=1}^{N_f} \tilde{\delta}\left(p_1 + p_2 - \sum_{n=1}^{N_f} p'_n\right) |\mathcal{M}|^2$$

where we have used the relativistic normalization of free particle states $\langle p|p\rangle = 2E\tilde{\delta}^{(3)}(0)$ and the fact that the S-matrix element can always be written as $S_{fi} = -i\delta(\sum p)\mathcal{M}$. Setting

$$\tilde{\delta}^{(3)}(0) = \frac{V}{(2\pi)^3}, \qquad \tilde{\delta}^{(4)}(0) = \frac{VT}{(2\pi)^4}$$

and writing the sum over final states as

$$\sum_{p'_n} \Longrightarrow \Pi_n \frac{V}{(2\pi)^3} \int d^3p'_n\,,$$

we obtain for $|S_{fi}|^2$

$$\frac{1}{4E_1E_2}\left(\frac{(2\pi)^3}{V}\right)^2 \frac{VT}{(2\pi)^4}\left(\frac{(2\pi)^3}{V}\right)^{N_f} \Pi_n^{N_f} \frac{V}{(2\pi)^3}\int \frac{d^3p'_n}{2E'_n}\tilde{\delta}\left(p_1 + p_2 - \sum_{n=1}^{N_f} p'_n\right) |\mathcal{M}|^2$$

$$= \left(\frac{T}{V}\right)\frac{(2\pi)^2}{4E_1E_2}\Pi_n^{N_f}\int \frac{d^3p'_n}{2E'_n}\tilde{\delta}\left(p_1 + p_2 - \sum_{n=1}^{N_f} p'_n\right) |\mathcal{M}|^2$$

Thus, we obtain

$$\sigma = \frac{(2\pi)^2}{4E_1E_2v}\Pi_n^{N_f}\int \frac{d^3p'_n}{2E'_n}\tilde{\delta}\left(p_1 + p_2 - \sum_{n=1}^{N_f} p'_n\right) |\mathcal{M}|^2$$

15.2 Decay rate in a scalar theory

Consider a real scalar field $\phi(x)$ interactiong with a complex scalar field $\chi(x)$ through the Lagrangian

$$\frac{1}{2}(\partial\phi)^2 + |\partial\chi|^2 - \frac{M^2}{2}\phi^2 - m|\chi|^2 - g\,\phi\,\chi^\dagger\,\chi\,. \tag{15.3}$$

This is a scalar analogue of a *"meson"* (ϕ) that can interact with or decay into a *"nucleon-antinucleon"* pair (χ, $\chi^\dagger$). *"Meson Number"* is clearly not conserved by the interaction term.

For the calculation scattering or decay amplitudes we need the S-matrix elements between initial and final states at temporal infinities $t \to \pm\infty$. For these states we make the assumption that they are *free states*, i.e. eigenstates of the free Hamiltonian. For the particular calculation of the *Decay Rate amplitude* for the process

$$\phi \to \chi + \chi^\dagger$$

we can take

$$\begin{aligned} |i\rangle &= (2\Omega(p))^{1/2}\; a^\dagger(p)|0\rangle \\ |f\rangle &= (4\omega(q_1)\omega(q_2))^{1/2}\; b^\dagger(q_1)\,c^\dagger(q_2)|0\rangle \end{aligned} \tag{15.4}$$

where the operators $a^\dagger$, $b^\dagger$, $c^\dagger$ refer to the free-field expansions

$$\begin{aligned} \phi_{in}(x) &= \int \tfrac{d^3p}{(2\pi)^{3/2}\sqrt{2\Omega}} \left(a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\right) \\ \chi_{out}(x) &= \int \tfrac{d^3q}{(2\pi)^{3/2}\sqrt{2\omega}} \left(b(q)e^{-iq\cdot x} + c^\dagger(q)e^{iq\cdot x}\right) \end{aligned} \tag{15.5}$$

and

$$\Omega(p) = \sqrt{\vec{p}^2 + M^2}, \quad \omega(q) = \sqrt{\vec{q}^2 + m^2}\,.$$

Next, we consider the *Interaction Picture* expression for the S-matrix element

$$S_{fi} = \langle f|T\left(e^{-i\int d^4x\,\mathcal{H}_{int}(x)}\right)|i\rangle \tag{15.6}$$

To lowest order in the coupling constant g we have

$$S_{fi} = -ig\int d^4x\,\langle f|\phi_{in}(x)\,\chi^\dagger_{out}(x)\,\chi_{out}(x)|i\rangle \tag{15.7}$$

First, using the expansions, we have that $\langle f|\phi_{in}(x)\,\chi^\dagger_{out}(x)\,\chi_{out}(x)|i\rangle$ is proportional to

$$\langle f|\left(a(p')e^{-ip'\cdot x}+a^\dagger(p')e^{ip'\cdot x}\right)\left(b(q_1')e^{-iq_1'\cdot x}+c^\dagger(q_1')e^{iq_1'\cdot x}\right)\left(b^\dagger(q_2')e^{iq_2'x}+c(q_2')e^{-iq_2'\cdot x}\right)|i\rangle$$

which is proportional to

$$\begin{aligned}&\langle 0|b(q_1)c(q_2)a(p')e^{-ip'\cdot x}b^\dagger(q_2')e^{iq_2'\cdot x}c^\dagger(q_1')e^{iq_1'\cdot x}a^\dagger(p)|0\rangle\\&\quad=\delta(\vec{q}_1{}'-\vec{q}_2)\delta(\vec{q}_2{}'-\vec{q}_1)\delta(\vec{p}\,'-\vec{p})\,e^{-i(p-q_1-q_2)\cdot x}\end{aligned}$$

Thus,

$$\begin{aligned}\langle f|\phi_{in}(x)\,\chi^\dagger_{out}(x)\,\chi_{out}(x)|i\rangle \;&=\; \sqrt{(2\omega(q_1))(2\omega(q_2))(2\Omega(p))}\\ \int\frac{d^3q_1'}{(2\pi)^{3/2}\sqrt{2\omega(q_1')}}&\int\frac{d^3q_2'}{(2\pi)^{3/2}\sqrt{2\omega(q_2')}}\int\frac{d^3p'}{(2\pi)^{3/2}\sqrt{2\Omega(p')}}\\ \delta(\vec{q}_1{}'-\vec{q}_2)\delta(\vec{q}_2{}'-\vec{q}_1)\delta(\vec{p}\,'-\vec{p})e^{-i(p-q_1-q_2)\cdot x}\;&=\;\frac{e^{-i(p-q_1-q_2)\cdot x}}{(2\pi)^{9/2}}\,.\end{aligned}$$

Then,

$$S_{fi}\;=\;-i\frac{g}{\sqrt{2\pi}}\int\frac{d^4x}{(2\pi)^4}\,e^{-i(p-q_1-q_2)\cdot x}\;=\;-\frac{ig}{\sqrt{2\pi}}\delta^{(4)}(p-q_1-q_2)\,. \tag{15.8}$$

The corresponding probability for the decay $\phi\to\chi+\chi^\dagger$ is

$$\mathcal{P}\;=\;\frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\,\langle i|i\rangle}$$

The norms of the initial and outgoing states are

$$\langle f|f\rangle\;=\;4\omega(q_1)\omega(q_2)\delta^{(3)}(0)\delta^{(3)}(0)\;=\;4\omega(q_1)\omega(q_2)\,\left(\tfrac{V}{(2\pi)^3}\right)^2$$

$$\langle i|i\rangle\;=\;2\Omega(p)\delta(0)\;=\;2\Omega(p)\,\tfrac{V}{(2\pi)^3}$$

where V stands for the three-dimensional space volume. For the initial meson at rest, we have

$$p\;=\;(M,\,0),\quad q_1\;=\;(E,\,\vec{q}),\;\; q_2\;=\;(E,\,-\vec{q}),\quad (E=\sqrt{\vec{q}^{\,2}+m^2})\,. \tag{15.9}$$

Then, we have[1]

$$\mathcal{P} \;=\; \frac{g^2}{(2\pi)}\frac{1}{(2M)(2E_1)(E_2)}\left(\frac{(2\pi)^3}{V}\right)^3\delta^{(4)}(p-q_1-q_2)\,\delta^{(4)}(0)$$

$$\frac{g^2}{(2M)(2E_1)(2E_2)}\frac{(2\pi)^8}{V^3}\frac{VT}{(2\pi)^4}\delta^{(4)}(p-q_1-q_2) \;=$$

$$=\;\frac{g^2}{8ME_1E_2}\frac{(2\pi)^4}{V^2}T\,\delta^{(4)}(p-q_1-q_2) \qquad (15.10)$$

The *Rate*, i.e. the probability per unit time is obtained just dividing $\mathcal{P}$ by T. Nevertheless, we should sum over the possible momenta of the final states

$$\sum_{\vec{q}_i} \to V\int\frac{d^3q_1}{(2\pi)^3}V\int\frac{d^3q_2}{(2\pi)^3}\,.$$

Thus, we have for the total rate

$$\Gamma \;=\; \frac{g^2}{2M}\int\frac{d^3q_1}{(2\pi)^3 2E_1}\int\frac{d^3q_2}{(2\pi)^3 2E_2}\,(2\pi)^4\delta(p-q_1-q_2)\,. \qquad (15.11)$$

For a general two-body decay, this generalizes to

$$\Gamma \;=\; \frac{1}{2M}\int\frac{d^3q_1}{(2\pi)^3 2E_1}\int\frac{d^3q_2}{(2\pi)^3 2E_2}\,|\mathcal{A}_{i\to f}|^2\,(2\pi)^4\delta(p-q_1-q_2)\,. \qquad (15.12)$$

The *lifetime* of the meson is the inverse of the total rate $\tau = \Gamma^{-1}$.

Going back to (15.11) we obtain

$$\Gamma \;=\; \frac{g^2}{16\pi M}\sqrt{1-\frac{4m^2}{M^2}}\,.$$

15.3 Scattering in the Yukawa theory

Consider the system of a Dirac fermion field $\Psi(x)$ and a real scalar field $\phi(x)$ coupled in terms of the Lagrangean

$$\mathcal{L} \;=\; i\overline{\Psi}\gamma^\mu\partial_\mu\Psi \,-\, m\overline{\Psi}\Psi \,+\, \frac{1}{2}(\partial\phi)^2 \,-\, \frac{\mu^2}{2}\phi^2 \,-\, \lambda\,\overline{\Psi}\Psi\,\phi\,. \qquad (15.13)$$

[1]Note that for the calculation of a general S-matrix element $-ig$ would be replaced by the *invariant amplitude* $\mathcal{A}_{i\to f}$.

This is a simplified version of the historic Yukawa theory that attempted to describe the strong interactions of nucleons through a coupling $\lambda\,\overline{\Psi}_n\gamma_5\Psi_n\,\pi$ to pseudoscalar mesons.

We consider as initial and final states

$$|i\rangle \;\equiv\; |\vec{q}_1, s_1;\, \vec{q}_2, s_2\rangle \;=\; (2\omega_1)^{1/2}(2\omega_2)^{1/2}\, a_{q_1}^{(s_1)\,\dagger}\, a_{q_2}^{(s_2)\,\dagger}\,|0\rangle$$

$$|f\rangle \;\equiv\; |\vec{q}_1{}', s_1';\, \vec{q}_2{}', s_2'\rangle \;=\; (2\omega_1')^{1/2}(2\omega_2')^{1/2}\, a_{q_1'}^{(s_1')\,\dagger}\, a_{q_2'}^{(s_2')\,\dagger}\,|0\rangle$$

Extracting the *Feynman rules* from the Lagrangian of the theory we can write the S-matrix element as

$$\mathcal{S}_{fi} \;=\; (-i\lambda)^2\delta(q_1{+}q_2{-}q_1'{-}q_2')\,\left(\frac{(\overline{u}^{(s_1)}(k_1)^{(s_1')}u(k_1'))(\overline{u}^{(s_2)}(q_2)u^{(s_2')}(q_2'))}{[(q_1-q_1')^2-\mu^2]}\right.$$
$$\left.-\frac{(\overline{u}^{(s_2)}(q_2)u^{(s_1)}(q_1'))(\overline{u}^{(s_1)}(q_1)u^{(s_2')}(q_2'))}{[(k_1-k_2')^2-\mu^2]}\right)$$

Summing over initial spins and averaging over final ones we obtain for the corresponding scattering probability

$$\frac{1}{4}\sum_{s'}|\mathcal{S}_{fi}|^2 \;=\; \frac{\lambda^4}{4}\delta(0)\delta(q_1{+}q_2{-}q_1'{-}q_2')\left|\frac{(\overline{u}^{(s_1)}(k_1)^{(s_1')}u(k_1'))(\overline{u}^{(s_2)}(q_2)u^{(s_2')}(q_2'))}{[(q_1-q_1')^2-\mu^2]}\right.$$
$$\left.-\frac{(\overline{u}^{(s_2)}(q_2)u^{(s_1)}(q_1'))(\overline{u}^{(s_1)}(q_1)u^{(s_2')}(q_2'))}{[(q_1-q_2')^2-\mu^2]}\right|^2$$

or

$$\frac{1}{4}\sum_{s_1,s_2,s_1',s_2'}|S_{fi}|^2 \;=\; \frac{\lambda^4}{4}\delta(0)\delta(q_1{+}q_2{-}q_1'{-}q_2')\sum_{s_1,s_2,s_1',s_2'}\left\{\frac{(\overline{u}(1')u(1))(\overline{u}(2')u(2))(\overline{u}(1)u(1'))(\overline{u}(2)u(2'}{[(q_1-q_1')^2-\mu^2]^2}\right.$$
$$+\frac{(\overline{u}(1')u(2))(\overline{u}(2')u(1))(\overline{u}(2)u(1'))(\overline{u}(1)u(2'))}{[(q_2-q_1')^2-\mu^2]^2}-\frac{(\overline{u}(1')u(1))(\overline{u}(2')u(2))(\overline{u}(2)u(1'))(\overline{u}(1)u(2'))}{[(q_1-q_1')^2-\mu^2][(q_2-q_1')^2-\mu^2]}$$
$$\left.-\frac{(\overline{u}(1)u(1'))(\overline{u}(2)u(2'))(\overline{u}(1')u(2))(\overline{u}(2')u(1))}{[(q_1-q_1')^2-\mu^2][(q_2-q_1')^2-\mu^2]}\right\}$$

But we have

$$\sum_{s_1,s_2,s_1',s_2'}(\overline{u}_A(1')u_A(1))(\overline{u}_B(2')u_B(2))(\overline{u}_C(1)u_C(1'))(\overline{u}_D(2)u_D(2')) \;=$$

$$\frac{1}{(2m)^4}\left(\gamma\cdot q_1'+m\right)_{CA}\left(\gamma\cdot q_1+m\right)_{AC}\left(\gamma\cdot q_2+m\right)_{BD}\left(\gamma\cdot q_2'+m\right)_{DB}\,=$$

$$\frac{1}{(2m)^4}Tr\left(\left(\gamma\cdot q_1'+m\right)\left(\gamma\cdot q_1+m\right)\right)Tr\left(\left(\gamma\cdot q_2+m\right)\left(\gamma\cdot q_2'+m\right)\right)$$

$$=\,\frac{1}{4m^4}\left((q_1'\cdot q_1)+m^2\right)\left((q_2'\cdot q_2)+m^2\right)$$

$$\sum_{s_1,s_2,s_1',s_2'}(\overline{u}(1')u(2))(\overline{u}(2')u(1))(\overline{u}(2)u(1'))(\overline{u}(1)u(2'))\,=\,\frac{1}{4m^4}\left((q_1'\cdot q_2)+m^2\right)\left((q_2'\cdot q_1)+m^2\right)$$

$$\sum_{s_1,s_2,s_1',s_2'}(\overline{u}_A(1')u_A(1))(\overline{u}_B(2')u_B(2))(\overline{u}_C(2)u_C(1'))(\overline{u}_D(1)u_D(2'))\,=$$

$$Tr\left(\left(\gamma\cdot q_1'+m\right)\left(\gamma\cdot q_1+m\right)\left(\gamma\cdot q_2'+m\right)\left(\gamma\cdot q_2+m\right)\right)\,=\,\frac{1}{4m^4}\left\{(q_1'\cdot q_2)(q_1\cdot q_2')\right.$$

$$+(q_2'\cdot q_2)(q_1'\cdot q_1)-(q_1\cdot q_2)(q_1'\cdot q_2')+m^4+$$

$$\left.m^2\left[(q_1'\cdot q_1)+(q_1\cdot q_2)+(q_1\cdot q_2')+(q_1'\cdot q_2)+(q_1'\cdot q_2')+(q_2'\cdot q_2)\right]\right\}$$

$$\sum_{s_1,s_2,s_1',s_2'}(\overline{u}(1)u(1'))(\overline{u}(2)u(2'))(\overline{u}(1')u(2))(\overline{u}(2')u(1))\,=\,\frac{1}{4m^4}\left\{(q_1'\cdot q_1)(q_2\cdot q_2')\right.$$

$$+(q_2'\cdot q_1)(q_1'\cdot q_2)-(q_2\cdot q_1)(q_1'\cdot q_2')+m^4+$$

$$\left.m^2\left[(q_1'\cdot q_2)+(q_1\cdot q_2)+(q_2\cdot q_2')+(q_1'\cdot q_1)+(q_1'\cdot q_2')+(q_2'\cdot q_1)\right]\right\}$$

We can write

$$\frac{1}{4}\sum_{s_1,s_2,s_1',s_2'}|S_{fi}|^2\,=\,\delta(0)\delta(q_1+q_2-q_1'-q_2')\,|\mathcal{M}|^2$$

with[2]

$$|\mathcal{M}|^2\,\equiv\,\frac{\lambda^4}{4}\left\{\frac{[4m^4-2m^2t+t^2/4]}{[t-\mu^2]^2}+\frac{[4m^4-2m^2u+u^2/4]}{[u-\mu^2]^2}\right.$$

[2]We introduce the *Mandelstam variables*

$$s=(q_1+q_2)^2,\;t=(q_1-q_1')^2,\quad u=(q_1-q_2')^2=(q_2-q_1')^2$$

$$(q_1\cdot q_2)=(q_1'\cdot q_2')=s/2-m^2$$

$$(q_2\cdot q_2')=(q_1\cdot q_1')=m^2-t/2$$

$$(q_1\cdot q_2')=(q_2\cdot q_1')=m^2-u/2$$

$$
-\frac{\left[8m^4+\frac{1}{2}(u^2+t^2-s^2)-4m^2(u+t-s)\right]}{[t^2-\mu^2][u^2-\mu^2]}\Bigg\}
$$