



ΠΑΝΕΠΙΣΤΗΜΙΟ ΙΩΑΝΝΙΝΩΝ
ΑΝΟΙΚΤΑ ΑΚΑΔΗΜΑΪΚΑ ΜΑΘΗΜΑΤΑ



Ψηφιακή Επεξεργασία Εικόνας

Φιλτράρισμα στο πεδίο των συχνοτήτων

Διδάσκων : Αναπληρωτής Καθηγητής Νίκου
Χριστόφορος



Ευρωπαϊκή Ένωση
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ΕΠΙΧΕΙΡΗΣΙΑΚΟ ΠΡΟΓΡΑΜΜΑ
ΕΚΠΑΙΔΕΥΣΗ ΚΑΙ ΔΙΑ ΒΙΟΥ ΜΑΘΗΣΗ
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EDUCATION AND LIFELONG LEARNING



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ΕΠΙΧΕΙΡΗΣΙΑΚΟ ΠΡΟΓΡΑΜΜΑ

ΥΠΟΥΡΓΕΙΟ ΠΑΙΔΕΙΑΣ & ΘΡΗΣΚΕΥΜΑΤΩΝ, ΠΟΛΙΤΙΣΜΟΥ & ΘΡΗΣΚΕΥΜΑΤΩΝ
ΕΙΔΙΚΗ ΥΠΗΡΕΣΙΑ ΔΙΑΧΕΙΡΙΣΗΣ
Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης

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Digital Image Processing

Filtering in the Frequency Domain (Circulant Matrices and Convolution)

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Toeplitz matrices

- Elements with constant value along the main diagonal and sub-diagonals.
- For a $N \times N$ matrix, its elements are determined by a $(2N-1)$ -length sequence $\{t_n \mid -(N-1) \leq n \leq N-1\}$

$$\mathbf{T}(m, n) = t_{m-n}$$

$$\mathbf{T} = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \dots & t_{-(N-1)} \\ t_1 & t_0 & t_{-1} & \dots & \vdots \\ t_2 & \dots & \dots & \dots & t_{-2} \\ \vdots & \dots & \dots & \dots & t_{-1} \\ t_{N-1} & \dots & t_2 & t_1 & t_0 \end{bmatrix}_{N \times N}$$

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Toeplitz matrices (cont.)

- Each row (column) is generated by a shift of the previous row (column).
 - The last element disappears.
 - A new element appears.

$$\mathbf{T}(m, n) = t_{m-n}$$

$$\mathbf{T} = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(N-1)} \\ t_1 & t_0 & t_{-1} & \cdots & \vdots \\ t_2 & \cdots & \cdots & \cdots & t_{-2} \\ \vdots & \cdots & \cdots & \cdots & t_{-1} \\ t_{N-1} & \cdots & t_2 & t_1 & t_0 \end{bmatrix}_{N \times N}$$

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Circulant matrices

- Elements with constant value along the main diagonal and sub-diagonals.
- For a $N \times N$ matrix, its elements are determined by a N -length sequence $\{c_n \mid 0 \leq n \leq N-1\}$

$$\mathbf{C}(m, n) = c_{(m-n) \bmod N}$$

$$\mathbf{C} = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \cdots & \vdots \\ c_{N-2} & \cdots & \cdots & \cdots & c_2 \\ \vdots & \cdots & \cdots & \cdots & c_1 \\ c_1 & c_2 & \cdots & c_{N-1} & c_0 \end{bmatrix}_{N \times N}$$

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Circulant matrices (cont.)

- Special case of a Toeplitz matrix.
- Each row (column) is generated by a **circular shift** (modulo N) of the previous row (column).

$$\mathbf{C}(m, n) = c_{(m-n) \bmod N}$$

$$\mathbf{C} = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \dots & \vdots \\ c_{N-2} & \dots & \dots & \dots & c_2 \\ \vdots & \dots & \dots & \dots & c_1 \\ c_1 & c_2 & \dots & c_{N-1} & c_0 \end{bmatrix}_{N \times N}$$

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Convolution by matrix-vector operations

- 1-D **linear convolution** between two discrete signals may be expressed as the product of a **Toeplitz matrix** constructed by the elements of one of the signals and a vector constructed by the elements of the other signal.
- 1-D **circular convolution** between two discrete signals may be expressed as the product of a **circulant matrix** constructed by the elements of one of the signals and a vector constructed by the elements of the other signal.
- Extension to 2D signals.

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1D linear convolution using Toeplitz matrices

$$f[n] = \{1, 2, 2\}, h[n] = \{1, -1\}, N_1 = 3, N_2 = 2$$

- The linear convolution $g[n] = f[n] * h[n]$ will be of length $N = N_1 + N_2 - 1 = 3 + 2 - 1 = 4$.
- We create a Toeplitz matrix \mathbf{H} from the elements of $h[n]$ (zero-padded if needed) with
 - $N=4$ lines (the length of the result).
 - $N_1=3$ columns (the length of $f[n]$).
 - The two signals may be interchanged.

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1D linear convolution using Toeplitz matrices (cont.)

$$f[n] = \{1, 2, 2\}, h[n] = \{1, -1\}, N_1 = 3, N_2 = 2$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}_{4 \times 3}$$

Length of the result = 4

Length of $f[n] = 3$

Zero-padded $h[n]$ in the first column

Notice that H is not circulant (e.g. a -1 appears in the second line which is not present in the first line).

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1D linear convolution using Toeplitz matrices (cont.)

$$f[n] = \{1, 2, 2\}, h[n] = \{1, -1\}, N_1 = 3, N_2 = 2$$

$$\mathbf{g} = \mathbf{H}\mathbf{f} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$$

$$g[n] = \{1, 1, 0, -2\}$$

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1D circular convolution using circulant matrices

$$f[n] = \{1, 2, 2\}, h[n] = \{1, -1\}, N_1 = 3, N_2 = 2$$

- The circular convolution $g[n] = f[n] \circledast h[n]$ will be of length $N = \max\{N_1, N_2\} = 3$.
- We create a circulant matrix \mathbf{H} from the elements of $h[n]$ (zero-padded if needed) of size $N \times N$.
 - The two signals may be interchanged.

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1D circular convolution using circulant matrices (cont.)

$$f[n] = \{1, 2, 2\}, h[n] = \{1, -1\}, N_1 = 3, N_2 = 2$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}_{3 \times 3}$$

Zero-padded $h[n]$ in the first column

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1D circular convolution using circulant matrices (cont.)

$$f[n] = \{1, 2, 2\}, h[n] = \{1, -1\}, N_1 = 3, N_2 = 2$$

$$\mathbf{g} = \mathbf{H}\mathbf{f} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$g[n] = \{-1, 1, 0\}$$

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Block matrices

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ A_{M1} & A_{M2} & \dots & A_{MN} \end{bmatrix}$$

- A_{ij} are matrices.
- If the structure of A , with respect to its sub-matrices, is Toeplitz (circulant) then matrix A is called **block-Toeplitz (block-circulant)**.
- If each individual A_{ij} is also a Toeplitz (circulant) matrix then A is called **doubly block-Toeplitz (doubly block-circulant)**.

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2D linear convolution using doubly block Toeplitz matrices

$$\begin{array}{c} m \\ \uparrow \\ \begin{array}{|c|c|c|} \hline 1 & 4 & 1 \\ \hline 2 & 5 & 3 \\ \hline \end{array} \\ \rightarrow n \\ f[m,n] \end{array}$$

$$M_1=2, N_1=3$$

$$\begin{array}{c} m \\ \uparrow \\ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & -1 \\ \hline \end{array} \\ \rightarrow n \\ h[m,n] \end{array}$$

$$M_2=2, N_2=2$$

The result will be of size $(M_1+M_2-1) \times (N_1+N_2-1) = 3 \times 4$

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2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\begin{array}{c} m \\ \uparrow \\ \begin{array}{|c|c|c|} \hline 1 & 4 & 1 \\ \hline 2 & 5 & 3 \\ \hline \end{array} \\ \rightarrow n \\ f[m,n]
 \end{array}$$

$$\begin{array}{c} m \\ \uparrow \\ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & -1 \\ \hline \end{array} \\ \rightarrow n \\ h[m,n]
 \end{array}$$

- At first, $h[m,n]$ is zero-padded to 3×4 (the size of the result).
- Then, for each row of $h[m,n]$, a Toeplitz matrix with 3 columns (the number of **columns** of $f[m,n]$) is constructed.

$$\begin{array}{c} m \\ \uparrow \\ \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ \hline 1 & -1 & 0 & 0 \\ \hline \end{array} \\ \rightarrow n \\ h[m,n]
 \end{array}$$

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2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\begin{array}{c} m \\ \uparrow \\ \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ \hline 1 & -1 & 0 & 0 \\ \hline \end{array} \\ \rightarrow n \\ h[m,n]
 \end{array}$$

- For each row of $h[m,n]$, a Toeplitz matrix with 3 columns (the number of columns of $f[m,n]$) is constructed.

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{ccc}
 1 & 4 & 1 \\
 2 & 5 & 3
 \end{array} \\
 \rightarrow n \\
 f[m,n]
 \end{array}$$

$$\begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{cc}
 1 & 1 \\
 1 & -1
 \end{array} \\
 \rightarrow n \\
 h[m,n]
 \end{array}$$

- Using matrices \mathbf{H}_1 , \mathbf{H}_2 and \mathbf{H}_3 as elements, a doubly block Toeplitz matrix \mathbf{H} is then constructed with 2 columns (the number of rows of $f[m,n]$).

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_3 \\ \mathbf{H}_2 & \mathbf{H}_1 \\ \mathbf{H}_3 & \mathbf{H}_2 \end{bmatrix}_{12 \times 6}$$

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2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{ccc}
 1 & 4 & 1 \\
 2 & 5 & 3
 \end{array} \\
 \rightarrow n \\
 f[m,n]
 \end{array}$$

$$\begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{cc}
 1 & 1 \\
 1 & -1
 \end{array} \\
 \rightarrow n \\
 h[m,n]
 \end{array}$$

- We now construct a vector from the elements of $f[m,n]$.

$$\mathbf{f} = \begin{bmatrix} 2 \\ 5 \\ 3 \\ 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} (2 \ 5 \ 3)^T \\ (1 \ 4 \ 1)^T \end{bmatrix}$$

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19 **2D linear convolution using doubly block Toeplitz matrices (cont.)**

$$\begin{matrix} m \\ \uparrow \\ \begin{matrix} 1 & 4 & 1 \\ 2 & 5 & 3 \end{matrix} \\ \rightarrow n \\ f[m,n] \end{matrix}$$

$$\begin{matrix} m \\ \uparrow \\ \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \\ \rightarrow n \\ h[m,n] \end{matrix}$$

$$\mathbf{g} = \mathbf{Hf} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_3 \\ \mathbf{H}_2 & \mathbf{H}_1 \\ \mathbf{H}_3 & \mathbf{H}_2 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 3 \\ 1 \\ 4 \\ 1 \end{bmatrix}$$

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20 **2D linear convolution using doubly block Toeplitz matrices (cont.)**

$$\mathbf{g} = \mathbf{Hf} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 3 \\ 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \\ 3 \\ -3 \\ 10 \\ 5 \\ 2 \\ -1 \\ 5 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} (2 & 3 & -2 & 3)^T \\ \hline (3 & 10 & 5 & 2)^T \\ \hline (1 & 5 & 5 & 1)^T \end{bmatrix}$$

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2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{|c|c|c|} \hline 1 & 4 & 1 \\ \hline 2 & 5 & 3 \\ \hline \end{array} \\
 \rightarrow n \\
 f[m,n]
 \end{array}
 *
 \begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & -1 \\ \hline \end{array} \\
 \rightarrow n \\
 h[m,n]
 \end{array}$$

$$=
 \begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{|c|c|c|c|} \hline 1 & 5 & 5 & 1 \\ \hline 3 & 10 & 5 & 2 \\ \hline 2 & 3 & -2 & 3 \\ \hline \end{array} \\
 \rightarrow n \\
 g[m,n]
 \end{array}
 \quad
 \mathbf{g} = \begin{bmatrix} (2 \ 3 \ -2 \ 3)^T \\ (3 \ 10 \ 5 \ 2)^T \\ (1 \ 5 \ 5 \ 1)^T \end{bmatrix}$$

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2D linear convolution using doubly block Toeplitz matrices (cont.)

Another example

$$\begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} \\
 \rightarrow n \\
 f[m,n] \\
 M_1=2, N_1=2
 \end{array}
 \quad
 \begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{|c|c|} \hline 1 & -1 \\ \hline \end{array} \\
 \rightarrow n \\
 h[m,n] \\
 M_2=1, N_2=2
 \end{array}$$

The result will be of size $(M_1+M_2-1) \times (N_1+N_2-1) = 2 \times 3$

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2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{|c|c|}
 \hline
 3 & 4 \\
 \hline
 1 & 2 \\
 \hline
 \end{array} \\
 \rightarrow n \\
 f[m,n]
 \end{array}$$

$$\begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{|c|c|}
 \hline
 1 & -1 \\
 \hline
 \end{array} \\
 \rightarrow n \\
 h[m,n]
 \end{array}$$

- At first, $h[m,n]$ is zero-padded to 2×3 (the size of the result).
- Then, for each line of $h[m,n]$, a Toeplitz matrix with 2 columns (the number of **columns** of $f[m,n]$) is constructed.

$$\begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{|c|c|c|}
 \hline
 0 & 0 & 0 \\
 \hline
 1 & -1 & 0 \\
 \hline
 \end{array} \\
 \rightarrow n \\
 h[m,n]
 \end{array}$$

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2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{|c|c|c|}
 \hline
 0 & 0 & 0 \\
 \hline
 1 & -1 & 0 \\
 \hline
 \end{array} \\
 \rightarrow n \\
 h[m,n]
 \end{array}$$

- For each row of $h[m,n]$, a Toeplitz matrix with 2 columns (the number of columns of $f[m,n]$) is constructed.

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{H}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{|c|c|}
 \hline
 3 & 4 \\
 \hline
 1 & 2 \\
 \hline
 \end{array} \\
 \rightarrow n \\
 f[m,n]
 \end{array}$$

$$\begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{|c|c|}
 \hline
 1 & -1 \\
 \hline
 \end{array} \\
 \rightarrow n \\
 h[m,n]
 \end{array}$$

- Using matrices \mathbf{H}_1 and \mathbf{H}_2 as elements, a doubly block Toeplitz matrix \mathbf{H} is then constructed with 2 columns (the number of rows of $f[m,n]$).

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_2 & \mathbf{H}_1 \end{bmatrix}_{6 \times 4}$$

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2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{|c|c|}
 \hline
 3 & 4 \\
 \hline
 1 & 2 \\
 \hline
 \end{array} \\
 \rightarrow n \\
 f[m,n]
 \end{array}$$

$$\begin{array}{c}
 m \\
 \uparrow \\
 \begin{array}{|c|c|}
 \hline
 1 & -1 \\
 \hline
 \end{array} \\
 \rightarrow n \\
 h[m,n]
 \end{array}$$

- We now construct a vector from the elements of $f[m,n]$.

$$\mathbf{f} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (1 \ 2)^T \\ (3 \ 4)^T \end{bmatrix}$$

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2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\begin{array}{c} m \\ \uparrow \\ \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} \\ \rightarrow n \\ f[m,n]
 \end{array}
 \qquad
 \begin{array}{c} m \\ \uparrow \\ \begin{array}{|c|c|} \hline 1 & -1 \\ \hline \end{array} \\ \rightarrow n \\ h[m,n]
 \end{array}$$

$$\mathbf{g} = \mathbf{Hf} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_2 & \mathbf{H}_1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

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2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\mathbf{g} = \mathbf{Hf} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{array} \right] \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 3 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} (1 \ 1 \ -2)^T \\ (3 \ 1 \ -4)^T \end{bmatrix}$$

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2D linear convolution using doubly block Toeplitz matrices (cont.)

$$\begin{array}{ccc}
 \begin{array}{c} m \\ \uparrow \\ \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} \\ \rightarrow n \\ f[m,n] \end{array} & * & \begin{array}{c} m \\ \uparrow \\ \begin{array}{|c|c|} \hline 1 & -1 \\ \hline \end{array} \\ \rightarrow n \\ h[m,n] \end{array} \\
 \\
 = & & \begin{array}{c} m \\ \uparrow \\ \begin{array}{|c|c|c|} \hline 3 & 1 & 4 \\ \hline 1 & 1 & -2 \\ \hline \end{array} \\ \rightarrow n \\ g[m,n] \end{array} \quad \mathbf{g} = \begin{bmatrix} (1 \ 1 \ -2)^T \\ \hline (3 \ 1 \ -4)^T \end{bmatrix}
 \end{array}$$

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2D circular convolution using doubly block circulant matrices

The circular convolution $g[m,n]=f[m,n] \circledast h[m,n]$
with $0 \leq m \leq M-1, 0 \leq n \leq N-1$,

may be expressed in matrix-vector form as:

$$\mathbf{g} = \mathbf{H}\mathbf{f}$$

where \mathbf{H} is a doubly block circulant matrix generated by $h[m,n]$ and \mathbf{f} is a vectorized form of $f[m,n]$.

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2D circular convolution using doubly block circulant matrices (cont.)

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_0 & \mathbf{H}_{M-1} & \mathbf{H}_{M-2} & \cdots & \mathbf{H}_1 \\ \mathbf{H}_1 & \mathbf{H}_0 & \mathbf{H}_{M-1} & \cdots & \mathbf{H}_2 \\ \mathbf{H}_2 & \mathbf{H}_1 & \mathbf{H}_0 & \cdots & \mathbf{H}_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathbf{H}_{M-1} & \mathbf{H}_{M-2} & \mathbf{H}_{M-3} & \cdots & \mathbf{H}_0 \end{bmatrix}$$

Each \mathbf{H}_j , for $j=1,..M$, is a circulant matrix with N columns (the number of **columns** of $f[m,n]$) generated from the elements of the j -th **row** of $h[m,n]$.

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2D circular convolution using doubly block circulant matrices (cont.)

$$\mathbf{H}_j = \begin{bmatrix} h[j,0] & h[j,N-1] & \cdots & h[j,1] \\ h[j,1] & h[j,0] & \cdots & h[j,2] \\ \vdots & \vdots & \vdots & \vdots \\ h[j,N-1] & h[j,N-2] & \cdots & h[j,0] \end{bmatrix}_{N \times N}$$

Each \mathbf{H}_j , for $j=1,..M$, is a $N \times N$ circulant matrix generated from the elements of the j -th **row** of $h[m,n]$.

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2D circular convolution using doubly block circulant matrices (cont.)

$$f[m,n] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$h[m,n] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\mathbf{H}_0 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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2D circular convolution using doubly block circulant matrices (cont.)

$$f[m,n] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$h[m,n] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\mathbf{g} = \mathbf{H}\mathbf{f} = \begin{bmatrix} \mathbf{H}_0 & \mathbf{H}_2 & \mathbf{H}_1 \\ \mathbf{H}_1 & \mathbf{H}_0 & \mathbf{H}_2 \\ \mathbf{H}_2 & \mathbf{H}_1 & \mathbf{H}_0 \end{bmatrix} \begin{bmatrix} (1 \ 2 \ 1)^T \\ (1 \ 3 \ -1)^T \\ (0 \ 1 \ 0)^T \end{bmatrix}$$

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2D circular convolution using doubly block circulant matrices (cont.)

$$\mathbf{g} = \mathbf{Hf} = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array} \right] \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \\ 3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 3 \\ -3 \\ 1 \\ 4 \\ -1 \\ -2 \end{bmatrix}$$

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2D circular convolution using doubly block circulant matrices (cont.)

$$\begin{array}{ccc} \begin{array}{c} m \\ \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 3 & -1 \\ 1 & 2 & 1 \end{array} \\ \xrightarrow{n} \\ f[m,n] \end{array} & \star & \begin{array}{c} m \\ \begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{array} \\ \xrightarrow{n} \\ h[m,n] \end{array} \\ \\ = & & \begin{array}{c} m \\ \begin{array}{ccc} 1 & 4 & -2 \\ 3 & 4 & -3 \\ 0 & 2 & -1 \end{array} \\ \xrightarrow{n} \\ g[m,n] \end{array} \quad \mathbf{g} = \begin{bmatrix} (0 \ 2 \ -1)^T \\ (3 \ 4 \ -3)^T \\ (1 \ 4 \ -2)^T \end{bmatrix} \end{array}$$

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Diagonalization of circulant matrices

Theorem: The columns of the inverse DFT matrix are eigenvectors of any circulant matrix. The corresponding eigenvalues are the DFT values of the signal generating the circulant matrix.

Proof: Let

$$w_N = e^{-j\frac{2\pi}{N}} \Leftrightarrow w_N^{nk} = e^{-j\frac{2\pi n}{N}k}$$

be the DFT basis elements of length N with:

$$0 \leq k \leq N-1, 0 \leq n \leq N-1,$$

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Diagonalization of circulant matrices (cont.)

We know that the DFT $F[k]$ of a 1D signal $f[n]$ may be expressed in matrix-vector form:

$$\mathbf{F} = \mathbf{A}\mathbf{f}$$

where

$$\mathbf{f} = [f[0], f[1], \dots, f[N-1]]^T, \mathbf{F} = [F[0], F[1], \dots, F[N-1]]^T$$

$$\mathbf{A} = \begin{bmatrix} (w_N^0)^0 & (w_N^0)^1 & (w_N^0)^2 & \dots & (w_N^0)^{N-1} \\ (w_N^1)^0 & (w_N^1)^1 & (w_N^1)^2 & \dots & (w_N^1)^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (w_N^{N-1})^0 & (w_N^{N-1})^1 & (w_N^{N-1})^2 & \dots & (w_N^{N-1})^{N-1} \end{bmatrix}$$

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Diagonalization of circulant matrices (cont.)

The inverse DFT is then expressed by:

$$\mathbf{f} = \mathbf{A}^{-1}\mathbf{F}$$

where

$$\mathbf{A}^{-1} = \frac{1}{N}(\mathbf{A}^*)^T = \frac{1}{N} \begin{bmatrix} (w_N^0)^0 & (w_N^0)^1 & (w_N^0)^2 & \dots & (w_N^0)^{N-1} \\ (w_N^1)^0 & (w_N^1)^1 & (w_N^1)^2 & \dots & (w_N^1)^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (w_N^{N-1})^0 & (w_N^{N-1})^1 & (w_N^{N-1})^2 & \dots & (w_N^{N-1})^{N-1} \end{bmatrix}^*$$

The theorem implies that any circulant matrix has eigenvectors the columns of \mathbf{A}^{-1} .

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Diagonalization of circulant matrices (cont.)

Let \mathbf{H} be a $N \times N$ circulant matrix generated by the 1D N -length signal $h[n]$, that is:

$$\mathbf{H}(m, n) = h[(m - n)_{\text{mod } N}] \triangleq h[m - n]_N$$

Let also α_k be the k -th column of the inverse DFT matrix \mathbf{A}^{-1} .

We will prove that α_k for any k , is an eigenvector of \mathbf{H} .

The m -th element of the vector $\mathbf{H}\alpha_k$, denoted by $[\mathbf{H}\alpha_k]_m$ is the result of the circular convolution of the signal $h[n]$ with α_k .

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Diagonalization of circulant matrices (cont.)

$$\begin{aligned}
 [\mathbf{H}\boldsymbol{\alpha}_k]_m &= \sum_{n=0}^{N-1} h[m-n]_N \alpha_k[n] = \frac{1}{N} \sum_{n=0}^{N-1} h[m-n]_N w_N^{-kn} \\
 &= \frac{1}{N} \sum_{l=m-n}^{m-(N-1)} h[l]_N w_N^{-k(m-l)} = \frac{1}{N} w_N^{-km} \sum_{l=m}^{m-(N-1)} h[l]_N w_N^{kl} \\
 &= \frac{1}{N} w_N^{-km} \left[\sum_{l=m-(N-1)}^{-1} h[l]_N w_N^{kl} + \sum_{l=0}^m h[l]_N w_N^{kl} \right]
 \end{aligned}$$

We will break it into two parts

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Diagonalization of circulant matrices (cont.)

$$\begin{aligned}
 &= \frac{1}{N} w_N^{-km} \left[\sum_{l=m-(N-1)}^{-1} h[l]_N w_N^{kl} + \sum_{l=0}^{N-1} h[l]_N w_N^{kl} - \sum_{l=m+1}^{N-1} h[l]_N w_N^{kl} \right] \\
 &\quad \text{Periodic with respect to } N. \\
 &= \frac{1}{N} w_N^{-km} \left[\sum_{l=\overset{\circ}{N}+m-(N-1)}^{\overset{\circ}{N}-1} h[l]_N w_N^{kl} + \sum_{l=0}^{N-1} h[l]_N w_N^{kl} - \sum_{l=m+1}^{N-1} h[l]_N w_N^{kl} \right] \\
 &= \frac{1}{N} w_N^{-km} \left[\sum_{l=m+1}^{N-1} h[l]_N w_N^{kl} + \sum_{l=0}^{N-1} h[l]_N w_N^{kl} - \sum_{l=m+1}^{N-1} h[l]_N w_N^{kl} \right] \Leftrightarrow
 \end{aligned}$$

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Diagonalization of circulant matrices (cont.)

$$[\mathbf{H}\mathbf{a}_k]_m = \frac{1}{N} w_N^{-km} \left[\sum_{l=0}^{N-1} h[l] w_N^{kl} \right] = H[k] [\mathbf{a}_k]_m$$

DFT of $h[n]$ at k .

This holds for any value of m . Therefore:

$$\mathbf{H}\mathbf{a}_k = H[k]\mathbf{a}_k$$

which means that \mathbf{a}_k , for any k , is an eigenvector of \mathbf{H} with corresponding eigenvalue the k -th element of $H[k]$, the DFT of the signal generating \mathbf{H} .

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Diagonalization of circulant matrices (cont.)

The above expression may be written in terms of the inverse DFT matrix:

$$\mathbf{H}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{\Lambda}$$

$$\mathbf{\Lambda} = \text{diag}\{H[0], H[1], \dots, H[N-1]\}$$

or equivalently: $\mathbf{\Lambda} = \mathbf{A}\mathbf{H}\mathbf{A}^{-1}$

Based on this diagonalization, we can prove the property between circular convolution and DFT.

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Diagonalization of circulant matrices (cont.)

$$\mathbf{g} = \mathbf{H}\mathbf{f} \Leftrightarrow \mathbf{g} = \mathbf{H}\mathbf{A}^{-1}\mathbf{A}\mathbf{f} \Leftrightarrow \mathbf{A}\mathbf{g} = \mathbf{A}\mathbf{H}\mathbf{A}^{-1}\mathbf{A}\mathbf{f} \Leftrightarrow \mathbf{G} = \mathbf{\Lambda}\mathbf{F}$$

$$\Leftrightarrow \begin{bmatrix} G[0] \\ G[0] \\ \vdots \\ G[N-1] \end{bmatrix} = \begin{bmatrix} H[0] & 0 & \dots & 0 \\ 0 & H[1] & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H[N-1] \end{bmatrix} \begin{bmatrix} F[0] \\ F[0] \\ \vdots \\ F[N-1] \end{bmatrix}$$

\downarrow \downarrow \downarrow
 DFT of $g[n]$ DFT of $h[n]$ DFT of $f[n]$

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Diagonalization of doubly block circulant matrices

- These properties may be generalized in 2D.
- We need to define the Kronecker product:

$$\mathbf{A} \in \mathbb{R}^{M \times N}, \mathbf{B} \in \mathbb{R}^{K \times L}$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1N}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2N}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1}\mathbf{B} & a_{M2}\mathbf{B} & \dots & a_{MN}\mathbf{B} \end{bmatrix}_{MK \times NL}$$

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Diagonalization of doubly block circulant matrices (cont.)

- The 2D signal $f[m,n]$, $0 \leq m \leq M-1$, $0 \leq n \leq N-1$, may be vectorized in lexicographic ordering (stacking one column after the other) to a vector:

$$\mathbf{f} \in \mathbb{R}^{MN \times 1}$$

- The DFT of $f[m,n]$, may be obtained in matrix-vector form:

$$F = (\mathbf{A} \otimes \mathbf{A})\mathbf{f}$$

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Diagonalization of doubly block circulant matrices (cont.)

Theorem: The columns of the inverse 2D DFT matrix

$$(\mathbf{A} \otimes \mathbf{A})^{-1}$$

are eigenvectors of any doubly block circulant matrix. The corresponding eigenvalues are the 2D DFT values of the 2D signal generating the doubly block circulant matrix:

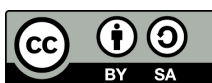
$$\mathbf{\Lambda} = (\mathbf{A} \otimes \mathbf{A})\mathbf{H}(\mathbf{A} \otimes \mathbf{A})^{-1}$$

Diagonal, containing the 2D DFT of $h[m,n]$ generating \mathbf{H}

Doubly block circulant

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Τέλος Ενότητας



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- Το παρόν εκπαιδευτικό υλικό έχει αναπτυχθεί στα πλαίσια του εκπαιδευτικού έργου του διδάσκοντα.
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